which, in turn, give (for example):

$$\left(\frac{X_1}{F_1}\right)_{f_2=0} = \frac{k_2 + k_3 - \omega^2 m_2}{\omega^4 m_1 m_2 - \omega^2\left((m_1+m_2)k_2 + m_1 k_3 + m_2 k_1\right) + \left(k_1 k_2 + k_2 k_3 + k_1 k_3\right)}$$

or, numerically,

$$= \frac{\left(1.2\times10^6 - \omega^2\right)}{\left(\omega^4 - 2.4\times10^6\omega^2 + 0.8\times10^{12}\right)}$$

Now. if we use the modal summation formula **(2.41)** together **with** the results obtained earlier, we can write

$$\alpha_{11} = \left(\frac{X_1}{F_1}\right) = \frac{(\phi_{11})^2}{\bar{\omega}_1^2 - \omega^2} + \frac{(\phi_{12})^2}{\bar{\omega}_2^2 - \omega^2}$$

or, numerically,

$$= \frac{0.5}{0.4\times10^6 - \omega^2} + \frac{0.5}{2\times10^6 - \omega^2}$$

which is equal to $\left(1.2\times10^6 - \omega^2\right)/\left(0.8\times10^{12} - 2.4\times10^6\,\omega^2 + \omega^4\right)$, as above.

The above characteristics of both the modal and response models of an undamped **MDOF** system form the basis of the corresponding data for the more general, damped, cases.

The following sections will examine the effects on these models of adding various types of damping, while a discussion of the presentation **MDOF** frequency response data is given in Section **2.10.**

2.5 MDOF SYSTEMS WITH PROPORTIONAL DAMPING

2.5.1 General Concept and Features of Proportional Damping

In approaching the more general case of damped systems, it is convenient to consider first a special type of damping which has the advantage of being particularly easy to include in our analysis. This type of damping is usually referred to as 'proportional' damping (for reasons which will be clear later) although this is a somewhat restrictive title. The particular advantage of using a proportional damping model in the analysis of structures is that the modes of such a structure are almost identical to those of the undamped version of the model. Specifically, the mode shapes are identical and the natural frequencies are very similar to those of the simpler undamped system. In fact, it is possible to derive the modal properties of a proportionally-damped system by analysing the undamped version in full and then making a correction for the presence of the damping. While this procedure is often used in the theoretical analysis of structures, it should be noted that it is only valid in the case of this special type or distribution of damping, which may not generally apply to the real structures studied in modal tests.

If we return to the general equation of motion for an MDOF system, equation (2.20), and add a viscous damping matrix [C], we obtain:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f\} \tag{2.45}$$

which is not so amenable to the type of solution followed in Section 2.4. A general solution will be presented in the next section, but here we shall examine the properties of this equation for the case where the damping matrix is directly proportional to the stiffness matrix; i.e. where

$$[C]=\beta[K] \tag{2.46}$$

(NOTE — It should be noted that this is not the only type of proportional damping — see below.)

In this case, it is clear that if we pre- and post-multiply the damping matrix by the eigenvector matrix for the undamped system, [Y], in just the same way as was done in equation (2.23) for the mass and stiffness matrices, then we shall find:

$$[\Psi]^T[C][\Psi]=\beta[k_r]=[c_r] \tag{2.47}$$

where the diagonal elements, c_r, represent the modal damping of the various modes of the system. The fact that this matrix is also diagonal means that the undamped system mode shapes are also those of the damped system, and this is a particular feature of this type of damping. This statement can easily be demonstrated by taking the general equation of motion above (2.45) and, for the case of no excitation, pre- and post-multiplying the whole equation by the eigenvector matrix, [Y]. We shall then find:

$$[m_r]\{\ddot{p}\}+[c_r]\{\dot{p}\}+[k_r]\{p\}=\{0\} \quad ; \quad \{p\}=[\Psi]^{-1}\{x\} \tag{2.48}$$

from which the r^{th} individual equation is:

$$m_r\ddot{p}_r + c_r\dot{p}_r + k_r p_r = 0 \tag{2.49}$$

which is clearly that of a single-degree-of-freedom system, or of a single mode of the system. This mode has a complex natural frequency with an imaginary (oscillatory) part of

$$\omega'_r = \bar{\omega}_r\sqrt{1-\zeta_r^2} \quad ; \quad \bar{\omega}_r^2 = \frac{k_r}{m_r} \quad ; \quad \zeta_r = \frac{c_r}{2\sqrt{k_r m_r}} = \frac{1}{2}\beta\bar{\omega}_r$$

and a real (decay) part of

$$a_r = \zeta_r\bar{\omega}_r = \frac{\beta}{2}$$

(using the notation introduced above for the SDOF analysis).

These characteristics carry over to the forced response analysis in which a simple extension of the steps detailed between equations **(2.35)** and (2.41) leads to the definition for the general receptance FRF as:

$$[\alpha(\omega)] = [K + i\omega C - \omega^2 M]^{-1}$$

or

$$\alpha_{jk}(\omega) = \sum_{r=1}^{N} \frac{(\psi_{jr})(\psi_{kr})}{(k_r - \omega^2 m_r) + i(\omega c_r)} \tag{2.50}$$

which has a very similar form to that for the undamped system except that now it becomes complex in the denominator as a result of the inclusion of damping.

2.5.2 General Forms of Proportional Damping

It may be seen from the above that other distributions of damping will bring about the same type of result and these are collectively included in the classification 'proportional damping'. In particular, if the damping matrix is proportional to the mass matrix, then exactly the same type of result ensues and, indeed, the usual definition of proportional damping is that the damping matrix [C] should be of the form:

$$[C] = \beta[K] + \gamma[M] \tag{2.51}$$

In this case, the damped system will have eigenvalues and eigenvectors as follows:

$$\omega_r' = \overline{\omega}_r\sqrt{1-\zeta_r^2} \quad ; \quad \zeta_r = \beta\overline{\omega}_r/2 + \gamma/2\overline{\omega}_r$$

and

$$[\Psi_{damped}] = [\Psi_{undamped}]$$

Distributions of damping of the type described above are sometimes, though not always, found to be plausible from a practical standpoint: the actual damping mechanisms are usually to be found in parallel with stiffness elements (for internal material or hysteresis damping) or with mass elements (for friction damping). There is a more general definition of the condition required for the damped system to possess the same mode shapes as its undamped counterpart, and that is:

$$\left([M]^{-1}[K]\right)\left([M]^{-1}[C]\right) = \left([M]^{-1}[C]\right)\left([M]^{-1}[K]\right) \tag{2.52}$$

although it is more difficult to make a direct physical interpretation of its form.

Finally, it can be noted that an identical treatment can be made of an MDOF system with proprtional hysteretic damping, producing the same essential results. If the general system equations of motion are expressed as:

$$[M]\{\ddot{x}\} + ([K + iD])\{x\} = \{f\} \tag{2.53}$$

and the hysteretic damping matrix $[D]$ is 'proportional', typically;

$$[D] = \beta[K] + \gamma[M] \tag{2.54}$$

then we find that the mode shapes for the damped system are again identical to those of the undamped system and that the eigenvalues take the complex form:

$$\lambda_r^2 = \overline{\omega}_r^2(1 + i\eta_r) \quad ; \quad \overline{\omega}_r^2 = k_r/m_r \quad ; \quad \eta_r = \beta + \gamma/\overline{\omega}_r^2 \tag{2.55}$$

Also, the general FRF expression is written:

$$\alpha_{jk}(\omega) = \sum_{r=1}^{N} \frac{(\psi_{jr})(\psi_{kr})}{(k_r - \omega^2 m_r) + i\eta_r k_r} \tag{2.56}$$

2.6 MDOF SYSTEMS WITH STRUCTURAL (HYSTERETIC) DAMPING — GENERAL CASE

2.6.1 Free Vibration Solution — Complex Modal Properties

The analysis in the previous section for proportionally-damped systems gives some insight into the characteristics of this more general description of practical structures. However, as was stated there, the case of proportional damping is a particular one which, although often justified in a theoretical analysis because it is realistic and also because of a lack of any more accurate model, does not apply to all cases. In our studies here, it is very important that we consider the most general case if we are to be able to interpret and analyse correctly the data we observe on real structures. These, after all, know nothing of our predilection for assuming proportionality in the distribution of damping. Thus, in the next two sections we consider the properties of systems with general damping elements, first of the hysteretic type, then viscous.

We start by writing the general equation of motion for an MDOF system with hysteretic damping and harmonic excitation (as it is this that we are working towards):

$$[M]\{\ddot{x}\} + [K]\{x\} + i[D]\{x\} = \{F\}e^{i\omega t} \tag{2.57}$$

Now, consider first the case where there is no excitation and assume a solution of the form:

$$\{x\} = \{X\}e^{i\lambda t} \tag{2.58}$$

where λ is allowed to be complex. Substituted into (2.57), this trial solution leads to a complex eigenproblem whose solution is in the form of two matrices (as for the earlier undamped case), containing the eigenvalues and eigenvectors. In this case, however, these matrices are both complex, meaning that each natural frequency and each mode shape is described in terms of complex quantities. We choose to write the r^{th} eigenvalue as

$$\lambda_r^2 = \omega_r^2(1 + i\eta_r) \tag{2.59}$$

where ω_r is the natural frequency and η_r is the damping loss factor for that mode. It is important to note that the natural frequency ω_r is not (necessarily) equal to the natural frequency of the undamped system, $\bar{\omega}_r$, as was the case for proportional hysteretic damping, although the two values will generally be very close in practice.

The complex mode shapes are at first more difficult to interpret but in fact what we find is that the amplitude of each DOF can be considered as having both a magnitude and a phase angle. This is only very slightly different from the undamped case as there we effectively have a magnitude at each point plus a phase angle which is either 0° or 180°, both of which can be completely described using real numbers. What the inclusion of general damping effects does is to generalise this particular feature of the mode shape data to a situation in which the phase may take any value, not only 0° and 180°. Further discussion of this feature is given in Section **2.9.**

This eigensolution can be seen to possess the same type of orthogonality properties as those demonstrated earlier for the undamped system and may be defined by the equations:

$$[\Psi]^T[M][\Psi]=[m_r] \quad ; \quad [\Psi]^T[K+iD][\Psi]=[k_r] \tag{2.60}$$

Again, the modal mass and stiffness parameters (now complex) depend upon the normalisation of the mode shape vectors for their magnitudes but always obey the relationship:

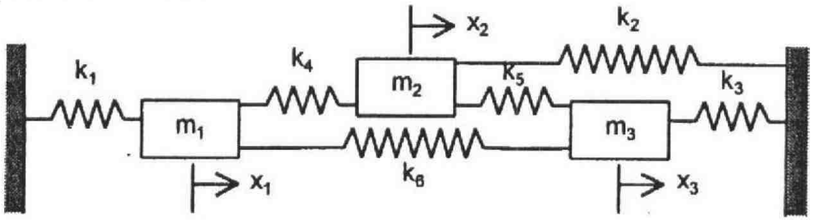
$$\lambda_r^2 = \frac{k_r}{m_r} \tag{2.61}$$

and here again we may define a set of mass-normalised eigenvectors as:

$$\{\phi\}_r = (m_r)^{-1/2}\{\psi\}_r \tag{2.62}$$

Numerical examples with structural damping

Some further numerical examples are included to illustrate the characteristics of more general damped systems, based on the following *3DOF* model:



Model 1

$m_1 = 0.5\,\text{kg} \qquad m_3 = 1.5\,\text{kg}$

$m_2 = 1.0\,\text{kg} \qquad k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 1.0\times 10^3\ \text{N/m}$

Case 1(a) — Undamped

$$\left[\overline{\omega}_r^2\right] = \begin{bmatrix} 950 & 0 & 0 \\ 0 & 3352 & 0 \\ 0 & 0 & 6698 \end{bmatrix} \quad ; \quad [\Phi] = \begin{bmatrix} 0.464 & -0.218 & -1.318 \\ 0.536 & -0.782 & 0.318 \\ 0.635 & 0.493 & 0.142 \end{bmatrix}$$

Case **1(b)** — Proportional structural damping $(d_j = 0.05k_j ; j = 1,6)$

$$\left[\lambda_r^2\right] = \begin{bmatrix} 950(1+0.05i) & 0 & 0 \\ 0 & 3352(1+0.05i) & 0 \\ 0 & 0 & 6698(1+0.05i) \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0.464(0^\circ) & 0.218(180^\circ) & 1.318(180^\circ) \\ 0.536(0^\circ) & 0.782(180^\circ) & 0.318(0^\circ) \\ 0.635(0^\circ) & 0.493(0^\circ) & 0.142(0^\circ) \end{bmatrix}$$

Case **1(c)** — Non-proportional structural damping $(d_1 = 0.3k_1, d_{2-6} = 0$, i.e. a single damper between m_1 and ground)

$$\left[\lambda_r^2\right] = \begin{bmatrix} 957(1+0.067i) & 0 & 0 \\ 0 & 3354(1+0.042i) & 0 \\ 0 & 0 & 6690(1+0.078i) \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0.463(-5.5^\circ) & 0.217(173^\circ) & 1.321(181^\circ) \\ 0.537(0^\circ) & 0.784(181^\circ) & 0.316(-6.7^\circ) \\ 0.636(1.0^\circ) & 0.492(-1.3^\circ) & 0.142(-3.1^\circ) \end{bmatrix}$$

NOTES:
(i) Each mode has a different damping factor.
(ii) All eigenvector arguments within 10° of 0° or 180° (i.e. the modes are almost 'real').

Model 2

$$m_1 = 1.0\ \text{kg} \qquad m_3 = 1.05\ \text{kg}$$

$$m_2 = 0.95\ \text{kg} \qquad k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 1.0 \times 10^3\ \text{N/m}$$

Case 2(a) — Undamped

$$\left[\overline{\omega}_r^2\right] = \begin{bmatrix} 999 & 0 & 0 \\ 0 & 3892 & 0 \\ 0 & 0 & 4124 \end{bmatrix} \quad ; \quad [\Phi] = \begin{bmatrix} 0.577 & -0.602 & 0.552 \\ 0.567 & -0.215 & -0.827 \\ 0.587 & 0.752 & 0.207 \end{bmatrix}$$

NOTE: this system has two close natural frequencies.

Case 2(b) — Proportional structural damping ($d_j = 0.05k_j$)

$$\left[\lambda_r^2\right] = \begin{bmatrix} 999(1+0.05i) & 0 & 0 \\ 0 & 3892(1+0.05i) & 0 \\ 0 & 0 & 4124(1+0.05i) \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0.577(0^\circ) & 0.602(180^\circ) & 0.552(0^\circ) \\ 0.567(0^\circ) & 0.215(180^\circ) & 0.827(180^\circ) \\ 0.587(0^\circ) & 0.752(0^\circ) & 0.207(0^\circ) \end{bmatrix}$$

Case 2(c) — Non-proportional structural damping ($d_1 = 0.3k_1, d_{2-6} = 0$)

$$\left[\lambda_r^2\right] = \begin{bmatrix} 1006(1+0.10i) & 0 & 0 \\ 0 & 3942(1+0.031i) & 0 \\ 0 & 0 & 4067(1+0.019i) \end{bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0.578(-4^\circ) & 0.851(162^\circ) & 0.685(40^\circ) \\ 0.569(2^\circ) & 0.570(101^\circ) & 1.019(176^\circ) \\ 0.588(2^\circ) & 0.848(12^\circ) & 0.560(-50^\circ) \end{bmatrix}$$

2.6.2 Forced Response Solution — FRF Characteristics

We turn next to the analysis of forced vibration for the particular case of harmonic excitation and response, for which the governing equation of motion is:

$$\left([K]+i[D]-\omega^2[M]\right)\{X\}e^{i\omega t}=\{F\}e^{i\omega t} \tag{2.63}$$

As before, a direct solution to this problem may be obtained by using the equations of motion to give:

$$\{X\}=\left([K]+i[D]-\omega^2[M]\right)^{-1}\{F\}=[\alpha(\omega)]\{F\} \tag{2.64}$$

but again this is very inefficient for numerical application and we shall make use of the same procedure as before by multiplying both sides of the equation by the eigenvectors. Starting with **(2.64)**, and following the same procedure as between equations **(2.38)** and **(2.40)**, we can write:

$$[\alpha(\omega)]=[\Phi]\left[\left(\lambda_r^2-\omega^2\right)\right]^{-1}[\Phi]^T \tag{2.65}$$

and from this full matrix equation we can extract any one FRF element, such as $\alpha_{jk}(\omega)$, and express it explicitly in a series form:

$$\alpha_{jk}(\omega)=\sum_{r=1}^{N}\frac{\left(\phi_{jr}\right)\left(\phi_{kr}\right)}{\omega_r^2-\omega^2+i\eta_r\,\omega_r^2} \tag{2.66}$$

which may also be rewritten in various alternative ways, such as:

$$\alpha_{jk}(\omega)=\sum_{r=1}^{N}\frac{\left(\psi_{jr}\right)\left(\psi_{kr}\right)}{m_r\left(\omega_r^2-\omega^2+i\eta_r\,\omega_r^2\right)}$$

or

$$\alpha_{jk}(\omega)=\sum_{r=1}^{N}\frac{{}_rA_{jk}}{\omega_r^2-\omega^2+i\eta_r\,\omega_r^2}$$

In these expressions, the numerator (as well as the denominator) is now complex as a result of the complexity of the eigenvectors. It is in this respect that the general damping case differs from that for proportional damping.