Three-Dimensional Kinetics of a Rigid Body

21

CHAPTER OBJECTIVES

- To introduce the methods for finding the moments of inertia and products of inertia of a body about various axes.
- To show how to apply the principles of work and energy and linear and angular momentum to a rigid body having three-dimensional motion.
- To develop and apply the equations of motion in three dimensions.
- To study gyroscopic and torque-free motion.

*21.1 Moments and Products of Inertia

When studying the planar kinetics of a body, it was necessary to introduce the moment of inertia I_G, which was computed about an axis perpendicular to the plane of motion and passing through the body's mass center G. For the kinetic analysis of three-dimensional motion it will sometimes be necessary to calculate six inertial quantities. These terms, called the moments and products of inertia, describe in a particular way the distribution of mass for a body relative to a given coordinate system that has a specified orientation and point of origin.

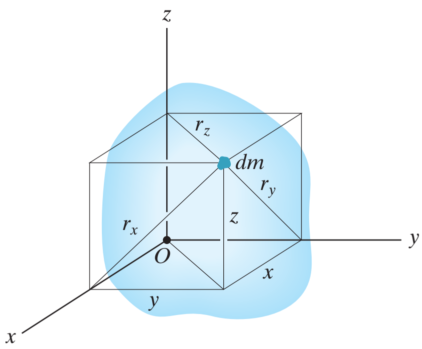

Fig. 21–1

Moment of Inertia. Consider the rigid body shown in Fig. 21–1. The *moment of inertia* for a differential element dm of the body about any one of the three coordinate axes is defined as the product of the mass of the element and the square of the shortest distance from the axis to the element. For example, as noted in the figure, $r_x = \sqrt{y^2 + z^2}$, so that the mass moment of inertia of the element about the x axis is

$$dI_{xx} = r_x^2\,dm = (y^2 + z^2)\,dm$$

The moment of inertia I_{xx} for the body can be determined by integrating this expression over the entire mass of the body. Hence, for each of the axes, we can write

$$\boxed{\begin{aligned} I_{xx} &= \int_m r_x^2\,dm = \int_m (y^2 + z^2)\,dm \\ I_{yy} &= \int_m r_y^2\,dm = \int_m (x^2 + z^2)\,dm \\ I_{zz} &= \int_m r_z^2\,dm = \int_m (x^2 + y^2)\,dm \end{aligned}} \tag{21–1}$$

Here it is seen that the moment of inertia is *always a positive quantity*, since it is the summation of the product of the mass dm, which is always positive, and the distances squared.

Product of Inertia. The *product of inertia* for a differential element dm with respect to a set of *two orthogonal planes* is defined as the product of the mass of the element and the perpendicular (or shortest) distances from the planes to the element. For example, this distance is x to the y–z plane and it is y to the x–z plane, Fig. 21–1. The product of inertia dI_{xy} for the element is therefore

$$dI_{xy} = xy\,dm$$

Note also that $dI_{yx} = dI_{xy}$. By integrating over the entire mass, the products of inertia of the body with respect to each combination of planes can be expressed as

21

$$\boxed{\begin{aligned} I_{xy} &= I_{yx} = \int_m xy\,dm \\ I_{yz} &= I_{zy} = \int_m yz\,dm \\ I_{xz} &= I_{zx} = \int_m xz\,dm \end{aligned}} \tag{21–2}$$

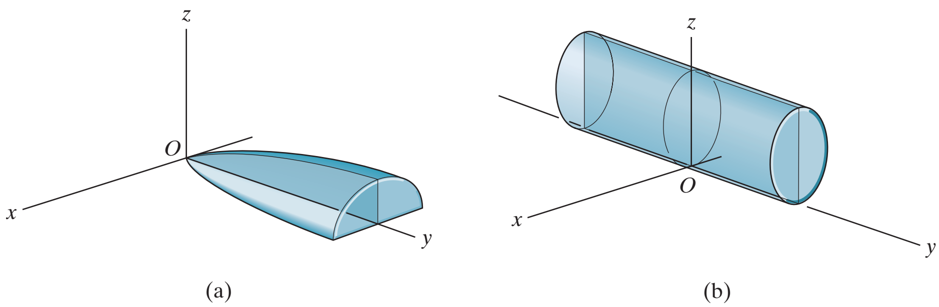

Fig. 21–2

Unlike the moment of inertia, which is always positive, the product of inertia may be positive, negative, or zero. The result depends on the algebraic signs of the two defining coordinates, which vary independently from one another. In particular, if either one or both of the orthogonal planes are *planes of symmetry* for the mass, the *product of inertia* with respect to these planes will be *zero*. In such cases, elements of mass will occur in *pairs* located on each side of the plane of symmetry. On one side of the plane the product of inertia for the element will be positive, while on the other side the product of inertia of the corresponding element will be negative, the sum therefore yielding zero. Examples of this are shown in Fig. 21–2. In the first case, Fig. 21–2*a*, the y–z plane is a plane of symmetry, and hence $I_{xy} = I_{xz} = 0$. Calculation of I_{yz} will yield a *positive* result, since all elements of mass are located using only positive y and z coordinates. For the cylinder, with the coordinate axes located as shown in Fig. 21–2*b*, the x–z and y–z planes are both planes of symmetry. Thus, $I_{xy} = I_{yz} = I_{zx} = 0$.

Parallel-Axis and Parallel-Plane Theorems. The techniques of integration used to determine the moment of inertia of a body were described in Sec. 17.1. Also discussed were methods to determine the moment of inertia of a composite body, i.e., a body that is composed of simpler segments, as tabulated on the inside back cover. In both of these cases the *parallel-axis theorem* is often used for the calculations. This theorem, which was developed in Sec. 17.1, allows us to transfer the moment of inertia of a body from an axis passing through its mass center G to a parallel axis passing through some other point. If G has coordinates x_G, y_G, z_G defined with respect to the x, y, z axes, Fig. 21–3, then the parallel-axis equations used to calculate the moments of inertia about the x, y, z axes are

$$\boxed{\begin{aligned} I_{xx} &= (I_{x'x'})_G + m(y_G^2 + z_G^2) \\ I_{yy} &= (I_{y'y'})_G + m(x_G^2 + z_G^2) \\ I_{zz} &= (I_{z'z'})_G + m(x_G^2 + y_G^2) \end{aligned}} \qquad (21\text{–}3)$$

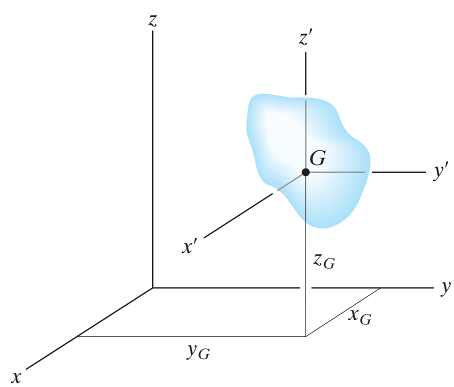

Fig. 21–3

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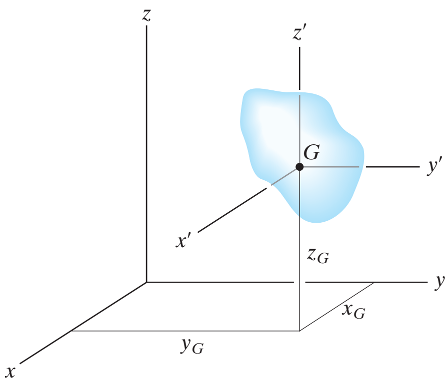


Fig. 21–3 (repeated)

The products of inertia of a composite body are computed in the same manner as the body's moments of inertia. Here, however, the *parallel-plane theorem* is important. This theorem is used to transfer the products of inertia of the body with respect to a set of three orthogonal planes passing through the body's mass center to a corresponding set of three parallel planes passing through some other point O. Defining the perpendicular distances between the planes as x_G, y_G and z_G, Fig. 21–3, the parallel-plane equations can be written as

$$\boxed{\begin{aligned} I_{xy} &= (I_{x'y'})_G + m x_G y_G \\ I_{yz} &= (I_{y'z'})_G + m y_G z_G \\ I_{zx} &= (I_{z'x'})_G + m z_G x_G \end{aligned}} \tag{21–4}$$

The derivation of these formulas is similar to that given for the parallel-axis equation, Sec. 17.1.

Inertia Tensor. The inertial properties of a body are therefore completely characterized by nine terms, six of which are independent of one another. This set of terms is defined using Eqs. 21–1 and 21–2 and can be written as

$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

This array is called an *inertia tensor*.* It has a unique set of values for a body when it is determined for each location of the origin O and orientation of the coordinate axes.

In general, for point O we can specify a unique axes inclination for which the products of inertia for the body are zero when computed with respect to these axes. When this is done, the inertia tensor is said to be "diagonalized" and may be written in the simplified form

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

The dynamics of the space shuttle while it orbits the earth can be predicted only if its moments and products of inertia are known relative to its mass center.

21

Here $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$ are termed the *principal moments of inertia* for the body, which are computed with respect to the *principal axes of inertia*. Of these three principal moments of inertia, one will be a maximum and another a minimum of the body's moment of inertia.

*The negative signs are here as a consequence of the development of angular momentum, Eqs. 21–10.

The mathematical determination of the directions of principal axes of inertia will not be discussed here (see Prob. 21–20). However, there are many cases in which the principal axes can be determined by inspection. From the previous discussion it was noted that if the coordinate axes are oriented such that *two* of the three orthogonal planes containing the axes are planes of *symmetry* for the body, then all the products of inertia for the body are zero with respect to these coordinate planes, and hence these coordinate axes are principal axes of inertia. For example, the x, y, z axes shown in Fig. 21–2b represent the principal axes of inertia for the cylinder at point O.

Moment of Inertia About an Arbitrary Axis. Consider the body shown in Fig. 21–4, where the nine elements of the inertia tensor have been determined with respect to the x, y, z axes having an origin at O. Here we wish to determine the moment of inertia of the body about the Oa axis, which has a direction defined by the unit vector $\mathbf{u}_a$. By definition $I_{Oa} = \int b^2\,dm$, where b is the *perpendicular distance* from dm to Oa. If the position of dm is located using $\mathbf{r}$, then $b = r\sin\theta$, which represents the *magnitude* of the cross product $\mathbf{u}_a \times \mathbf{r}$. Hence, the moment of inertia can be expressed as

$$I_{Oa} = \int_m |(\mathbf{u}_a \times \mathbf{r})|^2 dm = \int_m (\mathbf{u}_a \times \mathbf{r})\cdot(\mathbf{u}_a \times \mathbf{r})dm$$

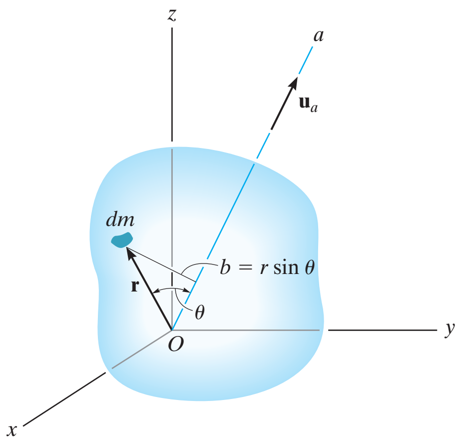


Fig. 21–4

Provided $\mathbf{u}_a = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{u}_a \times \mathbf{r} = (u_y z - u_z y)\mathbf{i} + (u_z x - u_x z)\mathbf{j} + (u_x y - u_y x)\mathbf{k}$. After substituting and performing the dot-product operation, the moment of inertia is

$$\begin{aligned} I_{Oa} &= \int_m [(u_y z - u_z y)^2 + (u_z x - u_x z)^2 + (u_x y - u_y x)^2]dm \\ &= u_x^2\int_m (y^2 + z^2)dm + u_y^2\int_m (z^2 + x^2)dm + u_z^2\int_m (x^2 + y^2)\,dm \\ &\qquad - 2u_x u_y\int_m xy\,dm - 2u_y u_z\int_m yz\,dm - 2u_z u_x\int_m zx\,dm \end{aligned}$$

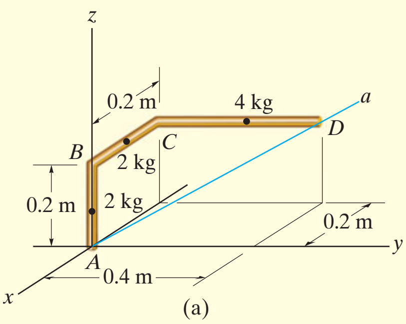
Recognizing the integrals to be the moments and products of inertia of the body, Eqs. 21–1 and 21–2, we have

$$\boxed{I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_x u_y - 2I_{yz}u_y u_z - 2I_{zx}u_z u_x} \qquad (21\text{–}5)$$

21

Thus, if the inertia tensor is specified for the x, y, z axes, the moment of inertia of the body about the inclined Oa axis can be found. For the calculation, the direction cosines u_x, u_y, u_z of the axes must be determined. These terms specify the cosines of the coordinate direction angles α, β, γ made between the positive Oa axis and the positive x, y, z axes, respectively (see Appendix C).

EXAMPLE 21.1



Determine the moment of inertia of the bent rod shown in Fig. 21–5*a* about the *Aa* axis. The mass of each of the three segments is given in the figure.

SOLUTION

Before applying Eq. 21–5, it is first necessary to determine the moments and products of inertia of the rod with respect to the *x, y, z* axes. This is done using the formula for the moment of inertia of a slender rod, $I = \frac{1}{12}ml^2$, and the parallel-axis and parallel-plane theorems, Eqs. 21–3 and 21–4. Dividing the rod into three parts and locating the mass center of each segment, Fig. 21–5*b*, we have

$$I_{xx} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + [0 + 2(0.2)^2] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((0.2)^2 + (0.2)^2)\right] = 0.480 \text{ kg}\cdot\text{m}^2$$

$$I_{yy} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2((-0.1)^2 + (0.2)^2)\right] + [0 + 4((-0.2)^2 + (0.2)^2)] = 0.453 \text{ kg}\cdot\text{m}^2$$

$$I_{zz} = [0 + 0] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2(-0.1)^2\right] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((-0.2)^2 + (0.2)^2)\right] = 0.400 \text{ kg}\cdot\text{m}^2$$

$$I_{xy} = [0 + 0] + [0 + 0] + [0 + 4(-0.2)(0.2)] = -0.160 \text{ kg}\cdot\text{m}^2$$

$$I_{yz} = [0 + 0] + [0 + 0] + [0 + 4(0.2)(0.2)] = 0.160 \text{ kg}\cdot\text{m}^2$$

$$I_{zx} = [0 + 0] + [0 + 2(0.2)(-0.1)] + [0 + 4(0.2)(-0.2)] = -0.200 \text{ kg}\cdot\text{m}^2$$

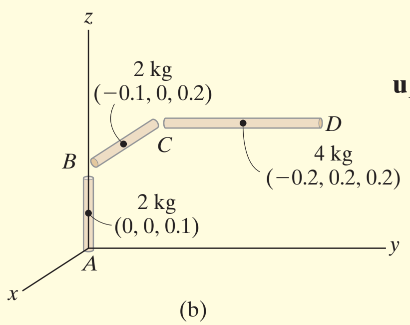


Fig. 21–5

The *Aa* axis is defined by the unit vector

$$\mathbf{u}_{Aa} = \frac{\mathbf{r}_D}{r_D} = \frac{-0.2\mathbf{i} + 0.4\mathbf{j} + 0.2\mathbf{k}}{\sqrt{(-0.2)^2 + (0.4)^2 + (0.2)^2}} = -0.408\mathbf{i} + 0.816\mathbf{j} + 0.408\mathbf{k}$$

Thus,

$$u_x = -0.408 \qquad u_y = 0.816 \qquad u_z = 0.408$$

Substituting these results into Eq. 21–5 yields

$$\begin{aligned} I_{Aa} &= I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x \\ &= 0.480(-0.408)^2 + (0.453)(0.816)^2 + 0.400(0.408)^2 \\ &\quad - 2(-0.160)(-0.408)(0.816) - 2(0.160)(0.816)(0.408) \\ &\quad - 2(-0.200)(0.408)(-0.408) \\ &= 0.169 \text{ kg}\cdot\text{m}^2 \end{aligned}$$

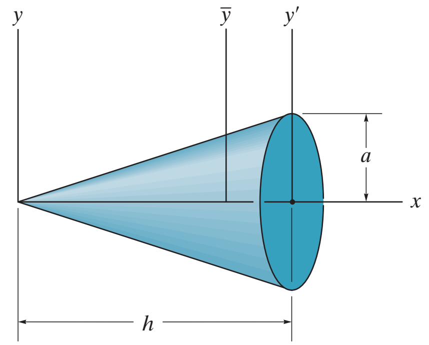
Ans.

21

PROBLEMS

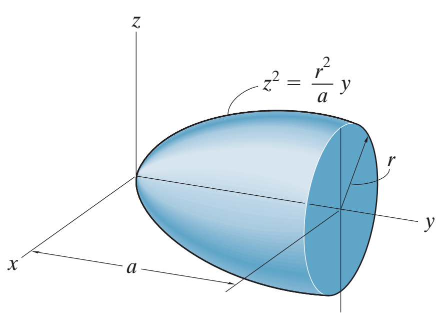
•21–1. Show that the sum of the moments of inertia of a body, $I_{xx} + I_{yy} + I_{zz}$, is independent of the orientation of the x, y, z axes and thus depends only on the location of its origin.

21–2. Determine the moment of inertia of the cone with respect to a vertical $\overline{y}$ axis that passes through the cone's center of mass. What is the moment of inertia about a parallel axis y' that passes through the diameter of the base of the cone? The cone has a mass m.



Prob. 21–2

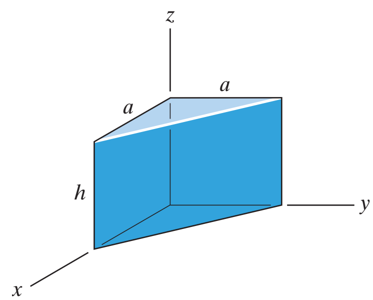
21–3. Determine the moments of inertia I_x and I_y of the paraboloid of revolution. The mass of the paraboloid is m.



Prob. 21–3

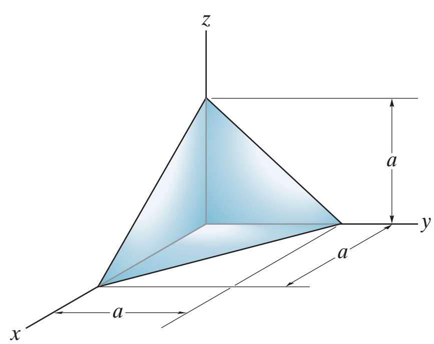
***21–4.** Determine by direct integration the product of inertia I_{yz} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.

•21–5. Determine by direct integration the product of inertia I_{xy} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.



Probs. 21–4/5

21–6. Determine the product of inertia I_{xy} for the homogeneous tetrahedron. The density of the material is ρ. Express the result in terms of the total mass m of the solid. *Suggestion:* Use a triangular element of thickness dz and then express dI_{xy} in terms of the size and mass of the element using the result of Prob. 21–5.



Prob. 21–6

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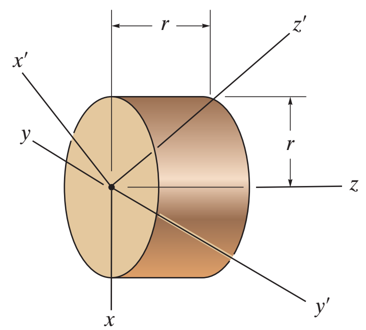
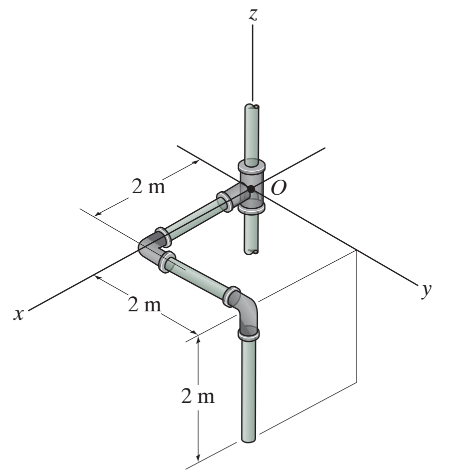
21–7. Determine the moments of inertia for the homogeneous cylinder of mass m about the x', y', z' axes.



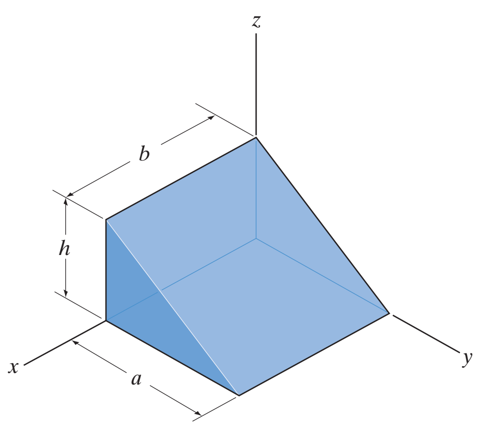
Prob. 21–7

•**21–9.** The slender rod has a mass per unit length of 6 kg/m. Determine its moments and products of inertia with respect to the x, y, z axes.



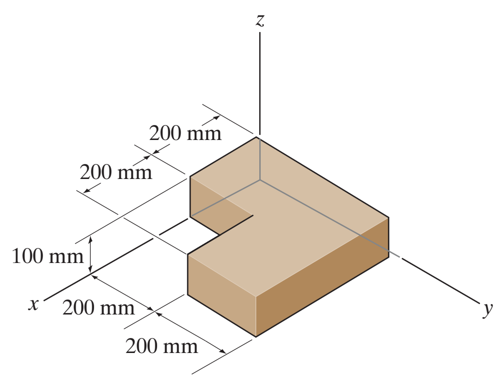
Prob. 21–9

*__21–8.__ Determine the product of inertia I_{xy} of the homogeneous triangular block. The material has a density of ρ. Express the result in terms of the total mass m of the block.



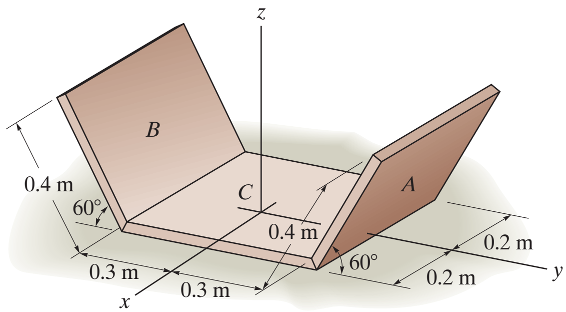
Prob. 21–8

21–10. Determine the products of inertia I_{xy}, I_{yz}, and I_{xz} of the homogeneous solid. The material has a density of 7.85 Mg/m^3.



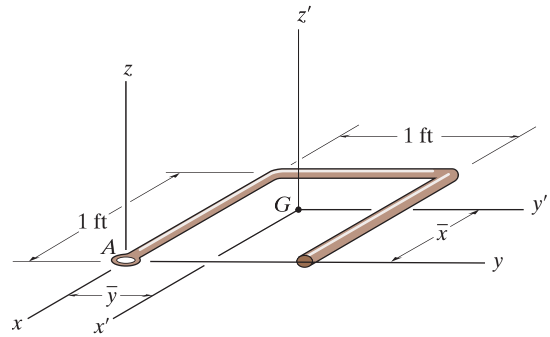
Prob. 21–10

21–11. The assembly consists of two thin plates A and B which have a mass of 3 kg each and a thin plate C which has a mass of 4.5 kg. Determine the moments of inertia I_x, I_y and I_z.



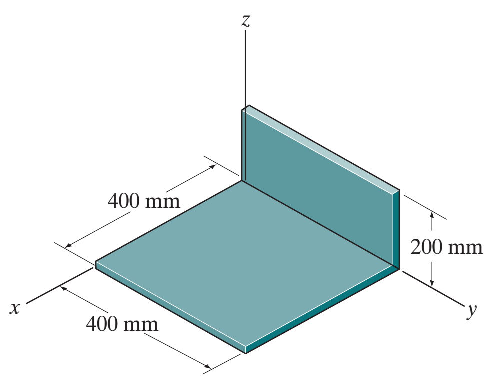
Prob. 21–11

•**21–13.** The bent rod has a weight of 1.5 lb/ft. Locate the center of gravity $G(\bar{x}, \bar{y})$ and determine the principal moments of inertia $I_{x'}$, $I_{y'}$, and $I_{z'}$ of the rod with respect to the x', y', z' axes.



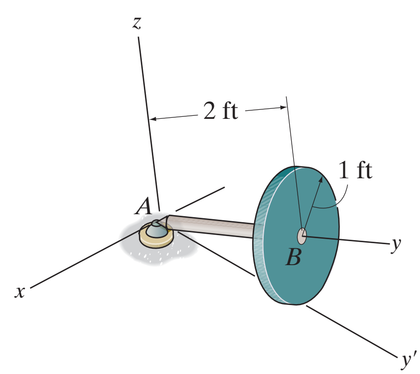
Prob. 21–13

*__21–12.__ Determine the products of inertia I_{xy}, I_{yz}, and I_{xz}, of the thin plate. The material has a density per unit area of 50 kg/m^2.



Prob. 21–12

21–14. The assembly consists of a 10-lb slender rod and a 30-lb thin circular disk. Determine its moment of inertia about the y' axis.



Prob. 21–14

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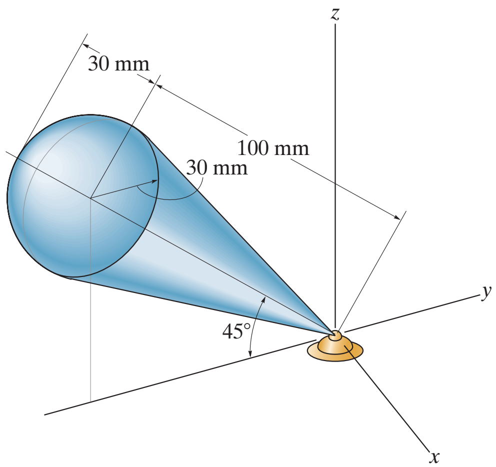
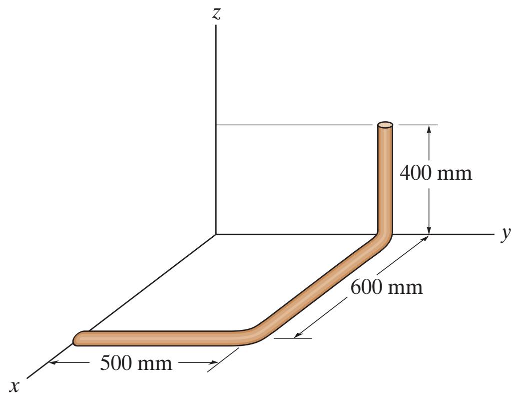
21–15. The top consists of a cone having a mass of 0.7 kg and a hemisphere of mass 0.2 kg. Determine the moment of inertia I_z when the top is in the position shown.



Prob. 21–15

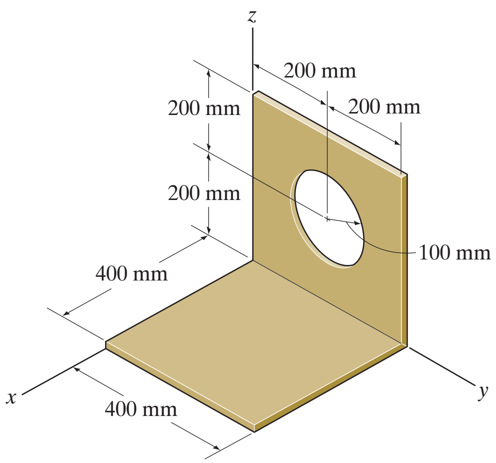
•21–17. Determine the product of inertia I_{xy} for the bent rod. The rod has a mass per unit length of 2 kg/m.

21–18. Determine the moments of inertia I_{xx}, I_{yy}, I_{zz} for the bent rod. The rod has a mass per unit length of 2 kg/m.



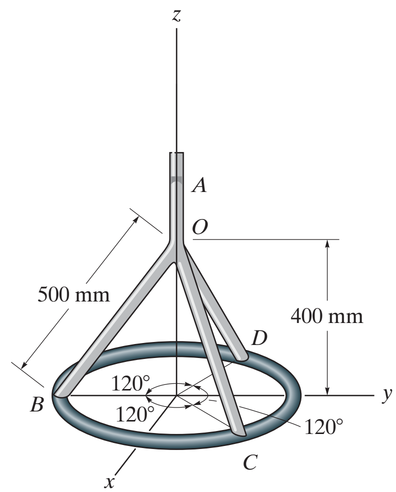
Probs. 21–17/18

***21–16.** Determine the products of inertia I_{xy}, I_{yz}, and I_{xz} of the thin plate. The material has a mass per unit area of 50 kg/m^2.



Prob. 21–16

21–19. Determine the moment of inertia of the rod-and-thin-ring assembly about the z axis. The rods and ring have a mass per unit length of 2 kg/m.



Prob. 21–19

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21.2 Angular Momentum

In this section we will develop the necessary equations used to determine the angular momentum of a rigid body about an arbitrary point. These equations will provide a means for developing both the principle of impulse and momentum and the equations of rotational motion for a rigid body.

Consider the rigid body in Fig. 21–6, which has a mass m and center of mass at G. The X, Y, Z coordinate system represents an inertial frame of reference, and hence, its axes are fixed or translate with a constant velocity. The angular momentum as measured from this reference will be determined relative to the arbitrary point A. The position vectors $\mathbf{r}_A$ and $\boldsymbol{\rho}_A$ are drawn from the origin of coordinates to point A and from A to the ith particle of the body. If the particle's mass is m_i, the angular momentum about point A is

$$(\mathbf{H}_A)_i = \boldsymbol{\rho}_A \times m_i \mathbf{v}_i$$

where $\mathbf{v}_i$ represents the particle's velocity measured from the X, Y, Z coordinate system. If the body has an angular velocity $\boldsymbol{\omega}$ at the instant considered, $\mathbf{v}_i$ may be related to the velocity of A by applying Eq. 20–7, i.e.,

$$\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$$

Thus,

$$\begin{aligned}(\mathbf{H}_A)_i &= \boldsymbol{\rho}_A \times m_i(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A) \\ &= (\boldsymbol{\rho}_A m_i) \times \mathbf{v}_A + \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) m_i\end{aligned}$$

Summing the moments of all the particles of the body requires an integration. Since $m_i \rightarrow dm$, we have

$$\mathbf{H}_A = \left(\int_m \boldsymbol{\rho}_A \, dm \right) \times \mathbf{v}_A + \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) dm \qquad (21\text{–}6)$$

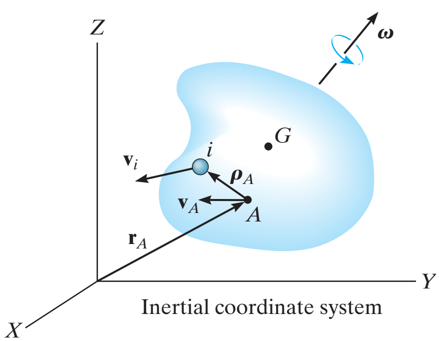
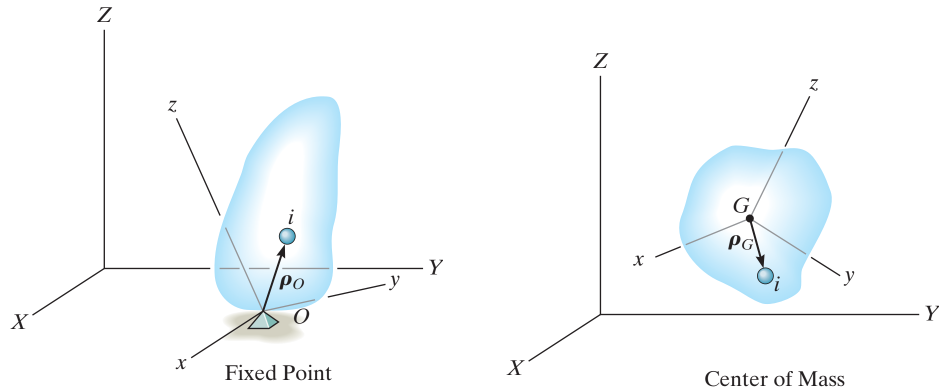


Fig. 21–6



Fixed Point
(a)
Center of Mass
(b)

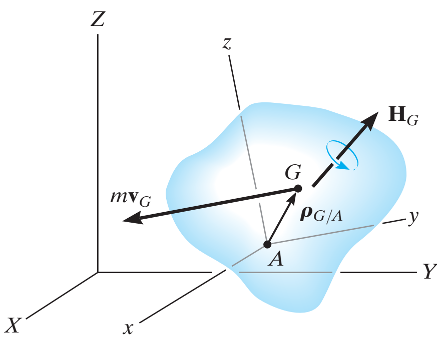
Fig. 21–7

Fixed Point *O*. If A becomes a *fixed point O* in the body, Fig. 21–7*a*, then $\mathbf{v}_A = \mathbf{0}$ and Eq. 21–6 reduces to

$$\boxed{\mathbf{H}_O = \int_m \boldsymbol{\rho}_O \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_O)\, dm} \tag{21–7}$$

Center of Mass *G*. If A is located at the *center of mass G* of the body, Fig. 21–7*b*, then $\int_m \boldsymbol{\rho}_A\, dm = \mathbf{0}$ and

$$\boxed{\mathbf{H}_G = \int_m \boldsymbol{\rho}_G \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_G)\, dm} \tag{21–8}$$



Arbitrary Point
(c)

Arbitrary Point *A*. In general, A can be a point other than O or G, Fig. 21–7*c*, in which case Eq. 21–6 may nevertheless be simplified to the following form (see Prob. 21–21).

$$\boxed{\mathbf{H}_A = \boldsymbol{\rho}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G} \tag{21–9}$$

Here the angular momentum consists of two parts—the moment of the linear momentum $m\mathbf{v}_G$ of the body about point A added (vectorially) to the angular momentum $\mathbf{H}_G$. Equation 21–9 can also be used to determine the angular momentum of the body about a fixed point O. The results, of course, will be the same as those found using the more convenient Eq. 21–7.

Rectangular Components of *H*. To make practical use of Eqs. 21–7 through 21–9, the angular momentum must be expressed in terms of its scalar components. For this purpose, it is convenient to choose a second set of x, y, z axes having an arbitrary orientation relative to the X, Y, Z axes, Fig. 21–7, and for a general formulation, note that Eqs. 21–7 and 21–8 are both of the form

$$\mathbf{H} = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})dm$$

Expressing $\mathbf{H}$, $\boldsymbol{\rho}$, and $\boldsymbol{\omega}$ in terms of x, y, z components, we have

$$H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = \int_m (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times [(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})]dm$$

Expanding the cross products and combining terms yields

$$\begin{aligned} H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = &\left[\omega_x \int_m (y^2 + z^2)dm - \omega_y \int_m xy\,dm - \omega_z \int_m xz\,dm\right]\mathbf{i} \\ &+ \left[-\omega_x \int_m xy\,dm + \omega_y \int_m (x^2 + z^2)dm - \omega_z \int_m yz\,dm\right]\mathbf{j} \\ &+ \left[-\omega_x \int_m zx\,dm - \omega_y \int_m yz\,dm + \omega_z \int_m (x^2 + y^2)dm\right]\mathbf{k} \end{aligned}$$

Equating the respective $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and recognizing that the integrals represent the moments and products of inertia, we obtain

$$\boxed{\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}} \qquad (21\text{–}10)$$

These equations can be simplified further if the x, y, z coordinate axes are oriented such that they become *principal axes of inertia* for the body at the point. When these axes are used, the products of inertia $I_{xy} = I_{yz} = I_{zx} = 0$, and if the principal moments of inertia about the x, y, z axes are represented as $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$, the three components of angular momentum become

$$\boxed{H_x = I_x\omega_x \qquad H_y = I_y\omega_y \qquad H_z = I_z\omega_z} \qquad (21\text{–}11)$$

21

The motion of the astronaut is controlled by use of small directional jets attached to his or her space suit. The impulses these jets provide must be carefully specified in order to prevent tumbling and loss of orientation.

Principle of Impulse and Momentum. Now that the formulation of the angular momentum for a body has been developed, the *principle of impulse and momentum*, as discussed in Sec. 19.2, can be used to solve kinetic problems which involve *force, velocity, and time*. For this case, the following two vector equations are available:

$$\boxed{m(\mathbf{v}_G)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m(\mathbf{v}_G)_2} \tag{21–12}$$

$$\boxed{(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2} \tag{21–13}$$

In three dimensions each vector term can be represented by three scalar components, and therefore a total of *six scalar equations* can be written. Three equations relate the linear impulse and momentum in the *x, y, z* directions, and the other three equations relate the body's angular impulse and momentum about the *x, y, z* axes. Before applying Eqs. 21–12 and 21–13 to the solution of problems, the material in Secs. 19.2 and 19.3 should be reviewed.

21.3 Kinetic Energy

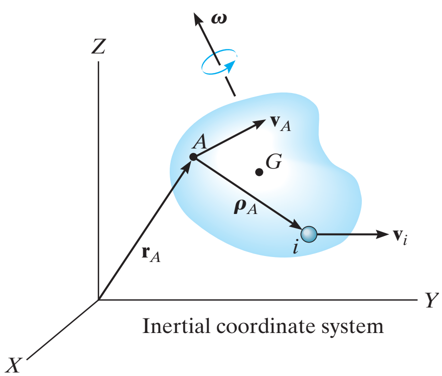


Fig. 21–8

In order to apply the principle of work and energy to solve problems involving general rigid body motion, it is first necessary to formulate expressions for the kinetic energy of the body. To do this, consider the rigid body shown in Fig. 21–8, which has a mass m and center of mass at G. The kinetic energy of the ith particle of the body having a mass m_i and velocity $\mathbf{v}_i$, measured relative to the inertial X, Y, Z frame of reference, is

$$T_i = \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i)$$

Provided the velocity of an arbitrary point A in the body is known, $\mathbf{v}_i$ can be related to $\mathbf{v}_A$ by the equation $\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$, where $\boldsymbol{\omega}$ is the angular velocity of the body, measured from the X, Y, Z coordinate system, and $\boldsymbol{\rho}_A$ is a position vector extending from A to i. Using this expression, the kinetic energy for the particle can be written as

$$\begin{aligned} T_i &= \tfrac{1}{2} m_i (\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A) \cdot (\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A) \\ &= \tfrac{1}{2}(\mathbf{v}_A \cdot \mathbf{v}_A) m_i + \mathbf{v}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) m_i + \tfrac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{\rho}_A) \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) m_i \end{aligned}$$

The kinetic energy for the entire body is obtained by summing the kinetic energies of all the particles of the body. This requires an integration. Since $m_i \rightarrow dm$, we get

$$T = \tfrac{1}{2} m(\mathbf{v}_A \cdot \mathbf{v}_A) + \mathbf{v}_A \cdot \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A\, dm\right) + \tfrac{1}{2} \int_m (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) dm$$

The last term on the right can be rewritten using the vector identity $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$, where $\mathbf{a} = \boldsymbol{\omega}$, $\mathbf{b} = \boldsymbol{\rho}_A$, and $\mathbf{c} = \boldsymbol{\omega} \times \boldsymbol{\rho}_A$. The final result is

$$T = \tfrac{1}{2}m(\mathbf{v}_A \cdot \mathbf{v}_A) + \mathbf{v}_A \cdot \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A \, dm\right) + \tfrac{1}{2}\boldsymbol{\omega} \cdot \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) dm \qquad (21\text{–}14)$$

This equation is rarely used because of the computations involving the integrals. Simplification occurs, however, if the reference point A is either a fixed point or the center of mass.

Fixed Point O. If A is a *fixed point O* in the body, Fig. 21–7*a*, then $\mathbf{v}_A = \mathbf{0}$, and using Eq. 21–7, we can express Eq. 21–14 as

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_O$$

If the x, y, z axes represent the principal axes of inertia for the body, then $\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$ and $\mathbf{H}_O = I_x \omega_x \mathbf{i} + I_y \omega_y \mathbf{j} + I_z \omega_z \mathbf{k}$. Substituting into the above equation and performing the dot-product operations yields

$$\boxed{T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2} \qquad (21\text{–}15)$$

Center of Mass G. If A is located at the *center of mass G* of the body, Fig. 21–7*b*, then $\int \boldsymbol{\rho}_A \, dm = \mathbf{0}$ and, using Eq. 21–8, we can write Eq. 21–14 as

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_G$$

In a manner similar to that for a fixed point, the last term on the right side may be represented in scalar form, in which case

$$\boxed{T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2} \qquad (21\text{–}16)$$

Here it is seen that the kinetic energy consists of two parts; namely, the translational kinetic energy of the mass center, $\tfrac{1}{2}mv_G^2$, and the body's rotational kinetic energy.

Principle of Work and Energy. Having formulated the kinetic energy for a body, the *principle of work and energy* can be applied to solve kinetics problems which involve *force, velocity, and displacement*. For this case only one scalar equation can be written for each body, namely,

$$\boxed{T_1 + \Sigma U_{1-2} = T_2} \qquad (21\text{–}17)$$

Before applying this equation, the material in Chapter 18 should be reviewed.

EXAMPLE 21.2

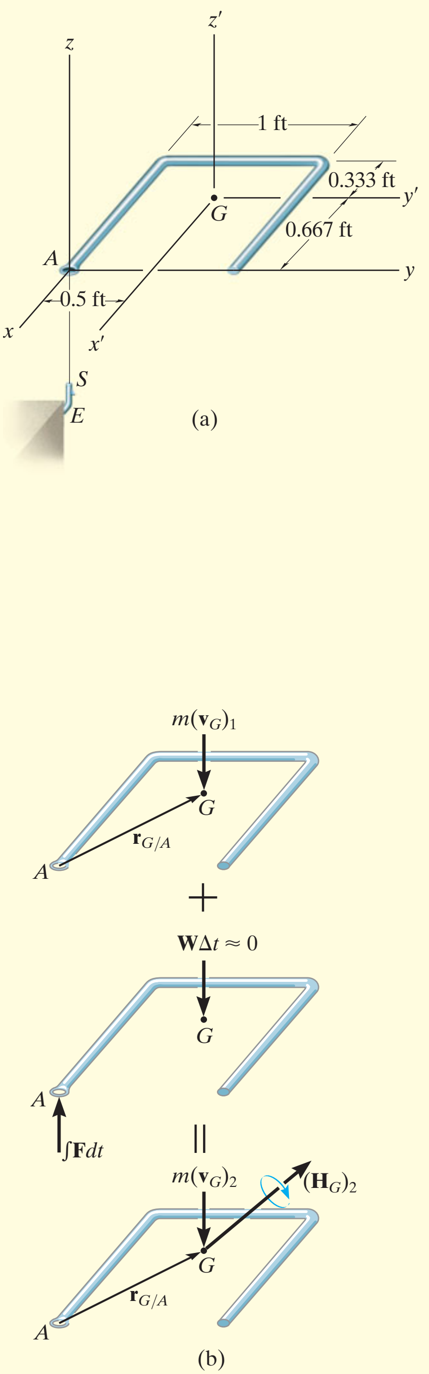


Fig. 21–9

The rod in Fig. 21–9a has a weight per unit length of 1.5 lb/ft. Determine its angular velocity just after the end A falls onto the hook at E. The hook provides a permanent connection for the rod due to the spring-lock mechanism S. Just before striking the hook the rod is falling downward with a speed $(v_G)_1 = 10$ ft/s.

SOLUTION

The principle of impulse and momentum will be used since impact occurs.

Impulse and Momentum Diagrams. Fig. 21–9b. During the short time Δt, the impulsive force $\mathbf{F}$ acting at A changes the momentum of the rod. (The impulse created by the rod's weight $\mathbf{W}$ during this time is small compared to $\int \mathbf{F}\,dt$, so that it can be neglected, i.e., the weight is a nonimpulsive force.) Hence, the angular momentum of the rod is *conserved* about point A since the moment of $\int \mathbf{F}\,dt$ about A is zero.

Conservation of Angular Momentum. Equation 21–9 must be used to find the angular momentum of the rod, since A does not become a *fixed point* until *after* the impulsive interaction with the hook. Thus, with reference to Fig. 21–9b, $(\mathbf{H}_A)_1 = (\mathbf{H}_A)_2$, or

$$\mathbf{r}_{G/A} \times m(\mathbf{v}_G)_1 = \mathbf{r}_{G/A} \times m(\mathbf{v}_G)_2 + (\mathbf{H}_G)_2 \qquad (1)$$

From Fig. 21–9a, $\mathbf{r}_{G/A} = \{-0.667\mathbf{i} + 0.5\mathbf{j}\}$ ft. Furthermore, the primed axes are principal axes of inertia for the rod because $I_{x'y'} = I_{x'z'} = I_{z'y'} = 0$. Hence, from Eqs. 21–11, $(\mathbf{H}_G)_2 = I_{x'}\omega_x\mathbf{i} + I_{y'}\omega_y\mathbf{j} + I_{z'}\omega_z\mathbf{k}$. The principal moments of inertia are $I_{x'} = 0.0272$ slug·ft^2, $I_{y'} = 0.0155$ slug·ft^2, $I_{z'} = 0.0427$ slug·ft^2 (see Prob. 21–13). Substituting into Eq. 1, we have

$$(-0.667\mathbf{i} + 0.5\mathbf{j}) \times \left[\left(\frac{4.5}{32.2}\right)(-10\mathbf{k})\right] = (-0.667\mathbf{i} + 0.5\mathbf{j}) \times \left[\left(\frac{4.5}{32.2}\right)(-v_G)_2\mathbf{k}\right]$$
$$+ 0.0272\omega_x\mathbf{i} + 0.0155\omega_y\mathbf{j} + 0.0427\omega_z\mathbf{k}$$

Expanding and equating the respective $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components yields

$$-0.699 = -0.0699(v_G)_2 + 0.0272\omega_x \qquad (2)$$
$$-0.932 = -0.0932(v_G)_2 + 0.0155\omega_y \qquad (3)$$
$$0 = 0.0427\omega_z \qquad (4)$$

Kinematics. There are four unknowns in the above equations; however, another equation may be obtained by relating $\boldsymbol{\omega}$ to $(\mathbf{v}_G)_2$ using *kinematics*. Since $\omega_z = 0$ (Eq. 4) and after impact the rod rotates about the fixed point A, Eq. 20–3 can be applied, in which case $(\mathbf{v}_G)_2 = \boldsymbol{\omega} \times \mathbf{r}_{G/A}$, or

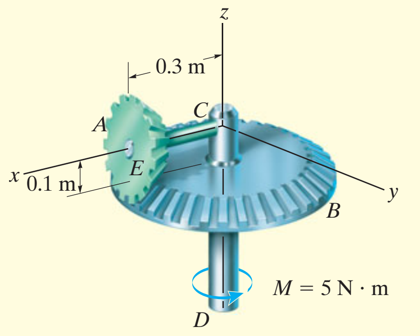
$$-(v_G)_2\mathbf{k} = (\omega_x\mathbf{i} + \omega_y\mathbf{j}) \times (-0.667\mathbf{i} + 0.5\mathbf{j})$$
$$-(v_G)_2 = 0.5\omega_x + 0.667\omega_y \qquad (5)$$

Solving Eqs. 2, 3 and 5 simultaneously yields

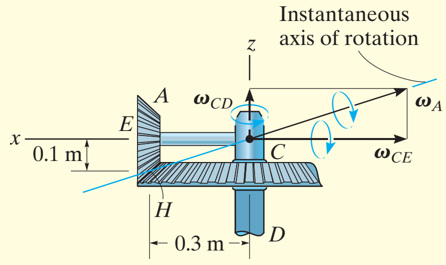
$$(\mathbf{v}_G)_2 = \{-8.41\mathbf{k}\}\text{ ft/s} \qquad \boldsymbol{\omega} = \{-4.09\mathbf{i} - 9.55\mathbf{j}\}\text{ rad/s} \quad \textit{Ans.}$$

EXAMPLE 21.3

A 5-N·m torque is applied to the vertical shaft *CD* shown in Fig. 21–10*a*, which allows the 10-kg gear *A* to turn freely about *CE*. Assuming that gear *A* starts from rest, determine the angular velocity of *CD* after it has turned two revolutions. Neglect the mass of shaft *CD* and axle *CE* and assume that gear *A* can be approximated by a thin disk. Gear *B* is fixed.



(a)



(b)

Fig. 21–10

SOLUTION

The principle of work and energy may be used for the solution. Why?

Work. If shaft *CD*, the axle *CE*, and gear *A* are considered as a system of connected bodies, only the applied torque **M** does work. For two revolutions of *CD*, this work is $\Sigma U_{1-2} = (5\text{ N}\cdot\text{m})(4\pi\text{ rad}) = 62.83\text{ J}$.

Kinetic Energy. Since the gear is initially at rest, its initial kinetic energy is zero. A kinematic diagram for the gear is shown in Fig. 21–10*b*. If the angular velocity of *CD* is taken as $\boldsymbol{\omega}_{CD}$, then the angular velocity of gear *A* is $\boldsymbol{\omega}_A = \boldsymbol{\omega}_{CD} + \boldsymbol{\omega}_{CE}$. The gear may be imagined as a portion of a massless extended body which is rotating about the *fixed point C*. The instantaneous axis of rotation for this body is along line *CH*, because both points *C* and *H* on the body (gear) have zero velocity and must therefore lie on this axis. This requires that the components $\boldsymbol{\omega}_{CD}$ and $\boldsymbol{\omega}_{CE}$ be related by the equation $\omega_{CD}/0.1\text{ m} = \omega_{CE}/0.3\text{ m}$ or $\omega_{CE} = 3\omega_{CD}$. Thus,

$$\boldsymbol{\omega}_A = -\omega_{CE}\mathbf{i} + \omega_{CD}\mathbf{k} = -3\omega_{CD}\mathbf{i} + \omega_{CD}\mathbf{k} \qquad (1)$$

The *x, y, z* axes in Fig. 21–10*a* represent *principal axes of inertia* at *C* for the gear. Since point *C* is a fixed point of rotation, Eq. 21–15 may be applied to determine the kinetic energy, i.e.,

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2 \qquad (2)$$

Using the parallel-axis theorem, the moments of inertia of the gear about point *C* are as follows:

$$I_x = \tfrac{1}{2}(10\text{ kg})(0.1\text{ m})^2 = 0.05\text{ kg}\cdot\text{m}^2$$

$$I_y = I_z = \tfrac{1}{4}(10\text{ kg})(0.1\text{ m})^2 + 10\text{ kg}(0.3\text{ m})^2 = 0.925\text{ kg}\cdot\text{m}^2$$

Since $\omega_x = -3\omega_{CD}$, $\omega_y = 0$, $\omega_z = \omega_{CD}$, Eq. 2 becomes

$$T_A = \tfrac{1}{2}(0.05)(-3\omega_{CD})^2 + 0 + \tfrac{1}{2}(0.925)(\omega_{CD})^2 = 0.6875\omega_{CD}^2$$

Principle of Work and Energy. Applying the principle of work and energy, we obtain

$$T_1 + \Sigma U_{1-2} = T_2$$

$$0 + 62.83 = 0.6875\omega_{CD}^2$$

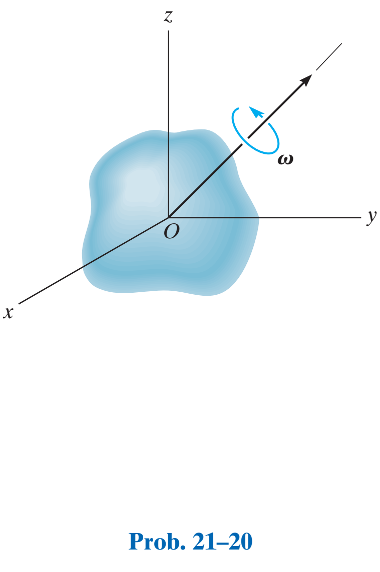
$$\omega_{CD} = 9.56\text{ rad/s} \qquad \textit{Ans.}$$

PROBLEMS

***21–20.** If a body contains *no planes of symmetry,* the principal moments of inertia can be determined mathematically. To show how this is done, consider the rigid body which is spinning with an angular velocity $\boldsymbol{\omega}$, directed along one of its principal axes of inertia. If the principal moment of inertia about this axis is I, the angular momentum can be expressed as $\mathbf{H} = I\boldsymbol{\omega} = I\omega_x\mathbf{i} + I\omega_y\mathbf{j} + I\omega_z\mathbf{k}$. The components of $\mathbf{H}$ may also be expressed by Eqs. 21–10, where the inertia tensor is assumed to be known. Equate the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of both expressions for $\mathbf{H}$ and consider ω_x, ω_y, and ω_z to be unknown. The solution of these three equations is obtained provided the determinant of the coefficients is zero. Show that this determinant, when expanded, yields the cubic equation

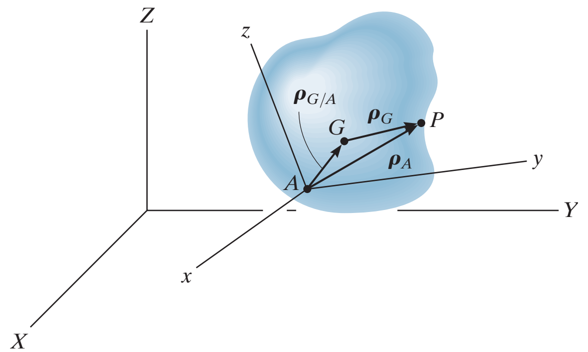
$$I^3 - (I_{xx} + I_{yy} + I_{zz})I^2 + (I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)I - (I_{xx}I_{yy}I_{zz} - 2I_{xy}I_{yz}I_{zx} - I_{xx}I_{yz}^2 - I_{yy}I_{zx}^2 - I_{zz}I_{xy}^2) = 0$$

The three positive roots of I, obtained from the solution of this equation, represent the principal moments of inertia I_x, I_y, and I_z.



Prob. 21–20

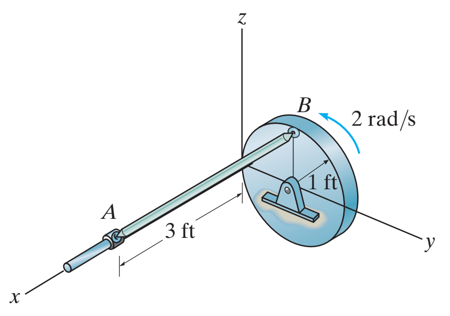
•21–21. Show that if the angular momentum of a body is determined with respect to an arbitrary point A, then $\mathbf{H}_A$ can be expressed by Eq. 21–9. This requires substituting $\boldsymbol{\rho}_A = \boldsymbol{\rho}_G + \boldsymbol{\rho}_{G/A}$ into Eq. 21–6 and expanding, noting that $\int \boldsymbol{\rho}_G \, dm = \mathbf{0}$ by definition of the mass center and $\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_{G/A}$.



Prob. 21–21

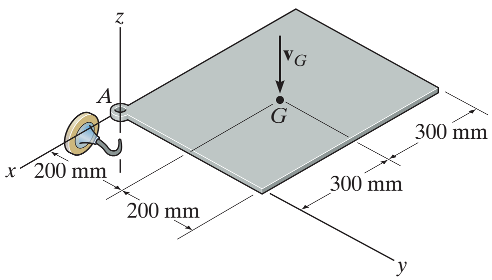
21–22. The 4-lb rod AB is attached to the disk and collar using ball-and-socket joints. If the disk has a constant angular velocity of 2 rad/s, determine the kinetic energy of the rod when it is in the position shown. Assume the angular velocity of the rod is directed perpendicular to the axis of the rod.

21–23. Determine the angular momentum of rod AB in Prob. 21–22 about its mass center at the instant shown. Assume the angular velocity of the rod is directed perpendicular to the axis of the rod.



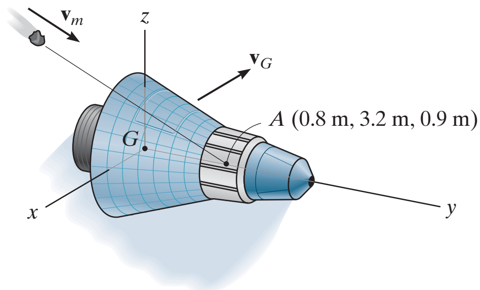
Probs. 21–22/23

*__21–24.__ The uniform thin plate has a mass of 15 kg. Just before its corner A strikes the hook, it is falling with a velocity of $\mathbf{v}_G = \{-5\mathbf{k}\}$ m/s with no rotational motion. Determine its angular velocity immediately after corner A strikes the hook without rebounding.



Prob. 21–24

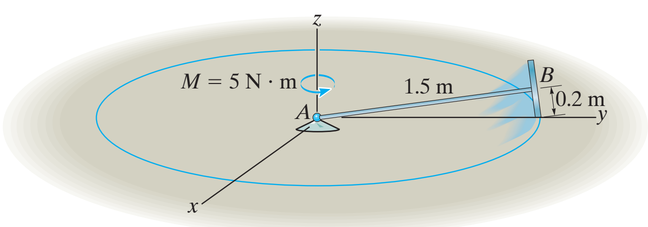
21–27. The space capsule has a mass of 5 Mg and the radii of gyration are $k_x = k_z = 1.30$ m and $k_y = 0.45$ m. If it travels with a velocity $\mathbf{v}_G = \{400\mathbf{j} + 200\mathbf{k}\}$ m/s, compute its angular velocity just after it is struck by a meteoroid having a mass of 0.80 kg and a velocity $\mathbf{v}_m = \{-300\mathbf{i} + 200\mathbf{j} - 150\mathbf{k}\}$ m/s. Assume that the meteoroid embeds itself into the capsule at point A and that the capsule initially has no angular velocity.



Prob. 21–27

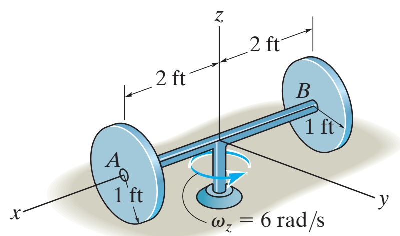
•**21–25.** The 5-kg disk is connected to the 3-kg slender rod. If the assembly is attached to a ball-and-socket joint at A and the 5-N·m couple moment is applied, determine the angular velocity of the rod about the z axis after the assembly has made two revolutions about the z axis starting from rest. The disk rolls without slipping.

21–26. The 5-kg disk is connected to the 3-kg slender rod. If the assembly is attached to a ball-and-socket joint at A and the 5-N·m couple moment gives it an angular velocity about the z axis of $\omega_z = 2$ rad/s, determine the magnitude of the angular momentum of the assembly about A.



Probs. 21–25/26

*__21–28.__ Each of the two disks has a weight of 10 lb. The axle AB weighs 3 lb. If the assembly rotates about the z axis at $\omega_z = 6$ rad/s, determine its angular momentum about the z axis and its kinetic energy. The disks roll without slipping.

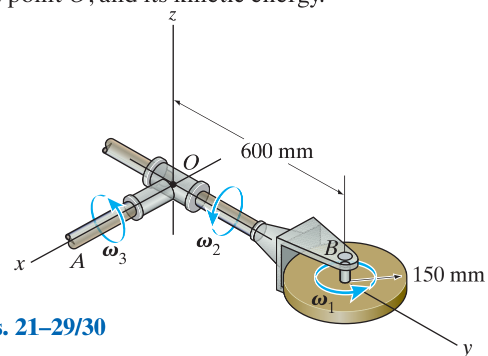


Prob. 21–28

21

•21–29. The 10-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 15$ rad/s. Simultaneously, arm OB and shaft OA rotate about their axes with constant angular velocities of $\omega_2 = 0$ and $\omega_3 = 6$ rad/s, respectively. Determine the angular momentum of the disk about point O, and its kinetic energy.

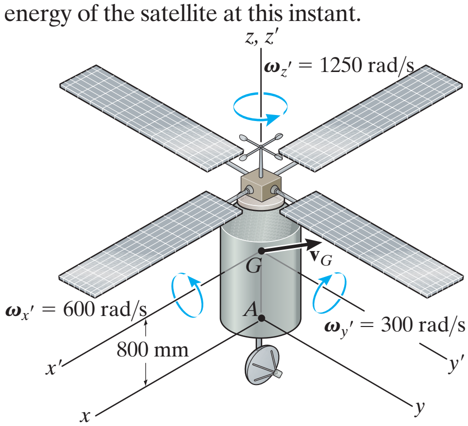
21–30. The 10-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 15$ rad/s. Simultaneously, arm OB and shaft OA rotate about their axes with constant angular velocities of $\omega_2 = 10$ rad/s and $\omega_3 = 6$ rad/s, respectively. Determine the angular momentum of the disk about point O, and its kinetic energy.



Probs. 21–29/30

21–31. The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the angular momentum of the satellite about point A at this instant.

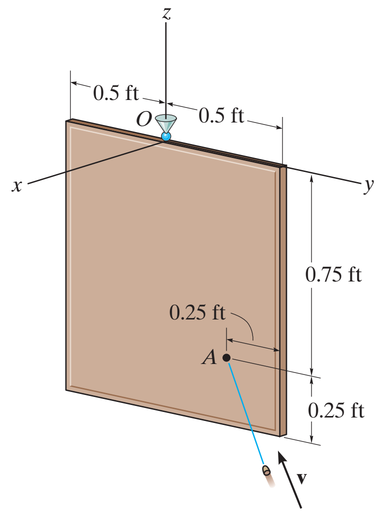
***21–32.** The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the kinetic energy of the satellite at this instant.



Probs. 21–31/32

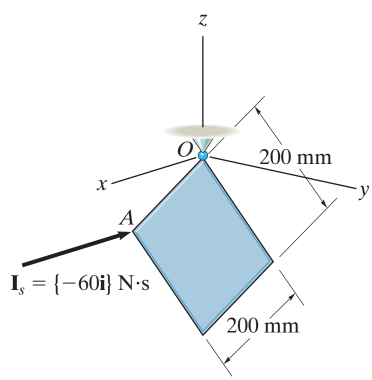
•21–33. The 25-lb thin plate is suspended from a ball-and-socket joint at O. A 0.2-lb projectile is fired with a velocity of $\mathbf{v} = \{-300\mathbf{i} - 250\mathbf{j} + 300\mathbf{k}\}$ ft/s into the plate and becomes embedded in the plate at point A. Determine the angular velocity of the plate just after impact and the axis about which it begins to rotate. Neglect the mass of the projectile after it embeds into the plate.

21–34. Solve Prob. 21–33 if the projectile emerges from the plate with a velocity of 275 ft/s in the same direction.



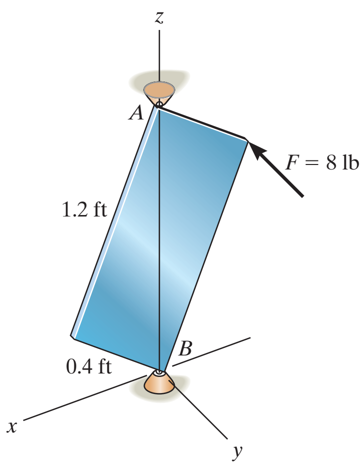
Probs. 21–33/34

21–35. A thin plate, having a mass of 4 kg, is suspended from one of its corners by a ball-and-socket joint O. If a stone strikes the plate perpendicular to its surface at an adjacent corner A with an impulse of $\mathbf{I}_s = \{-60\mathbf{i}\}$ N·s, determine the instantaneous axis of rotation for the plate and the impulse created at O.



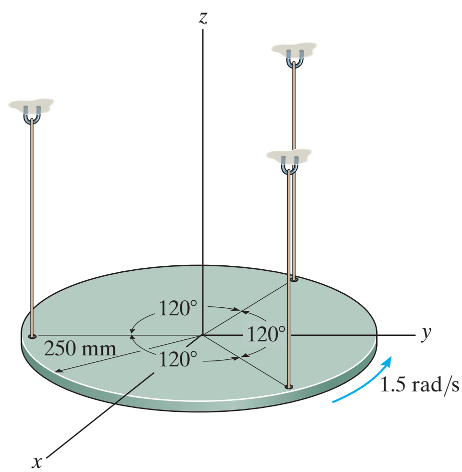
Prob. 21–35

***21–36.** The 15-lb plate is subjected to a force $F = 8$ lb which is always directed perpendicular to the face of the plate. If the plate is originally at rest, determine its angular velocity after it has rotated one revolution (360°). The plate is supported by ball-and-socket joints at A and B.



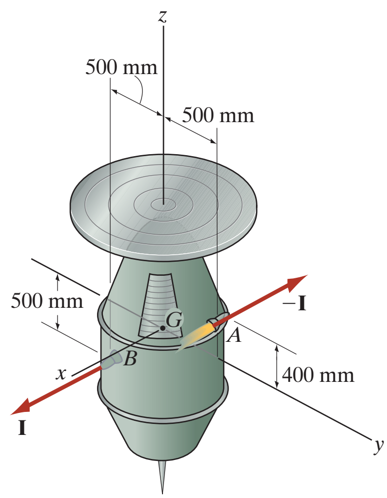
Prob. 21–36

•21–37. The plate has a mass of 10 kg and is suspended from parallel cords. If the plate has an angular velocity of 1.5 rad/s about the z axis at the instant shown, determine how high the center of the plate rises at the instant the plate momentarily stops swinging.



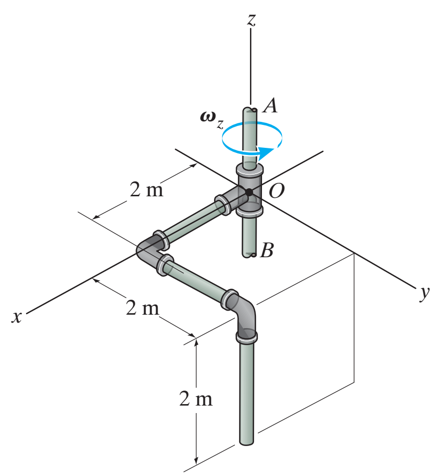
Prob. 21–37

21–38. The satellite has a mass of 200 kg and radii of gyration of $k_x = k_y = 400$ mm and $k_z = 250$ mm. When it is not rotating, the two small jets A and B are ignited simultaneously, and each jet provides an impulse of $I = 1000\ \text{N} \cdot \text{s}$ on the satellite. Determine the satellite's angular velocity immediately after the ignition.



Prob. 21–38

21–39. The bent rod has a mass per unit length of 6 kg/m, and its moments and products of inertia have been calculated in Prob. 21–9. If shaft AB rotates with a constant angular velocity of $\omega_z = 6$ rad/s, determine the angular momentum of the rod about point O, and the kinetic energy of the rod.



Prob. 21–39

*21.4 Equations of Motion

Having become familiar with the techniques used to describe both the inertial properties and the angular momentum of a body, we can now write the equations which describe the motion of the body in their most useful forms.

Equations of Translational Motion. The *translational motion* of a body is defined in terms of the acceleration of the body's mass center, which is measured from an inertial *X, Y, Z* reference. The equation of translational motion for the body can be written in vector form as

$$\Sigma \mathbf{F} = m\mathbf{a}_G \tag{21–18}$$

or by the three scalar equations

$$\boxed{\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma F_z &= m(a_G)_z \end{aligned}} \tag{21–19}$$

Here, $\Sigma \mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$ represents the sum of all the external forces acting on the body.

Equations of Rotational Motion. In Sec. 15.6, we developed Eq. 15–17, namely,

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \tag{21–20}$$

which states that the sum of the moments of all the external forces acting on a system of particles (contained in a rigid body) about a fixed point O is equal to the time rate of change of the total angular momentum of the body about point O. When moments of the external forces acting on the particles are summed about the system's *mass center* G, one again obtains the same simple form of Eq. 21–20, relating the moment summation $\Sigma \mathbf{M}_G$ to the angular momentum $\mathbf{H}_G$. To show this, consider the system of particles in Fig. 21–11, where X, Y, Z represents an inertial frame of reference and the x, y, z axes, with origin at G, *translate* with respect to this frame. In general, G is *accelerating*, so by definition the translating frame is *not* an inertial reference. The angular momentum of the ith particle with respect to this frame is, however,

$$(\mathbf{H}_i)_G = \mathbf{r}_{i/G} \times m_i \mathbf{v}_{i/G}$$

where $\mathbf{r}_{i/G}$ and $\mathbf{v}_{i/G}$ represent the position and velocity of the ith particle with respect to G. Taking the time derivative we have

$$(\dot{\mathbf{H}}_i)_G = \dot{\mathbf{r}}_{i/G} \times m_i \mathbf{v}_{i/G} + \mathbf{r}_{i/G} \times m_i \dot{\mathbf{v}}_{i/G}$$

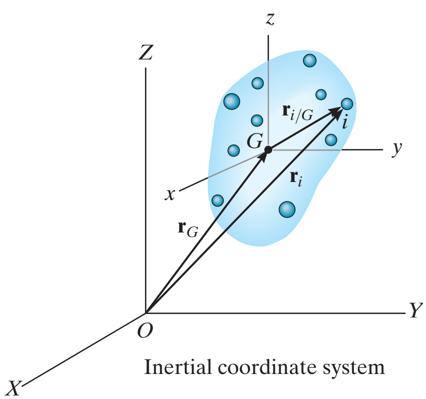


Fig. 21–11

21

By definition, $\mathbf{v}_{i/G} = \dot{\mathbf{r}}_{i/G}$. Thus, the first term on the right side is zero since the cross product of the same vectors is zero. Also, $\mathbf{a}_{i/G} = \dot{\mathbf{v}}_{i/G}$, so that

$$(\dot{\mathbf{H}}_i)_G = (\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Similar expressions can be written for the other particles of the body. When the results are summed, we get

$$\dot{\mathbf{H}}_G = \Sigma(\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Here $\dot{\mathbf{H}}_G$ is the time rate of change of the total angular momentum of the body computed about point G.

The relative acceleration for the ith particle is defined by the equation $\mathbf{a}_{i/G} = \mathbf{a}_i - \mathbf{a}_G$, where $\mathbf{a}_i$ and $\mathbf{a}_G$ represent, respectively, the accelerations of the ith particle and point G measured with respect to the *inertial frame of reference*. Substituting and expanding, using the distributive property of the vector cross product, yields

$$\dot{\mathbf{H}}_G = \Sigma(\mathbf{r}_{i/G} \times m_i \mathbf{a}_i) - (\Sigma m_i \mathbf{r}_{i/G}) \times \mathbf{a}_G$$

By definition of the mass center, the sum $(\Sigma m_i \mathbf{r}_{i/G}) = (\Sigma m_i)\bar{\mathbf{r}}$ is equal to zero, since the position vector $\bar{\mathbf{r}}$ relative to G is zero. Hence, the last term in the above equation is zero. Using the equation of motion, the product $m_i \mathbf{a}_i$ can be replaced by the resultant *external force* $\mathbf{F}_i$ acting on the ith particle. Denoting $\Sigma \mathbf{M}_G = \Sigma(\mathbf{r}_{i/G} \times \mathbf{F}_i)$, the final result can be written as

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{21–21}$$

The rotational equation of motion for the body will now be developed from either Eq. 21–20 or 21–21. In this regard, the scalar components of the angular momentum $\mathbf{H}_O$ or $\mathbf{H}_G$ are defined by Eqs. 21–10 or, if principal axes of inertia are used either at point O or G, by Eqs. 21–11. If these components are computed about x, y, z axes that are *rotating* with an angular velocity $\boldsymbol{\Omega}$ that is *different* from the body's angular velocity $\boldsymbol{\omega}$, then the time derivative $\dot{\mathbf{H}} = d\mathbf{H}/dt$, as used in Eqs. 21–20 and 21–21, must account for the rotation of the x, y, z axes as measured from the inertial X, Y, Z axes. This requires application of Eq. 20–6, in which case Eqs. 21–20 and 21–21 become

$$\begin{aligned} \Sigma \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \\ \Sigma \mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \end{aligned} \tag{21–22}$$

Here $(\dot{\mathbf{H}})_{xyz}$ is the time rate of change of $\mathbf{H}$ measured from the x, y, z reference.

There are three ways in which one can define the motion of the x, y, z axes. Obviously, motion of this reference should be chosen so that it will yield the simplest set of moment equations for the solution of a particular problem.

x, y, z Axes Having Motion $\mathbf{\Omega} = \mathbf{0}$. If the body has general motion, the *x, y, z* axes can be chosen with origin at *G*, such that the axes only *translate* relative to the inertial *X, Y, Z* frame of reference. Doing this simplifies Eq. 21–22, since $\mathbf{\Omega} = \mathbf{0}$. However, the body may have a rotation $\boldsymbol{\omega}$ about these axes, and therefore the moments and products of inertia of the body would have to be expressed as *functions of time*. In most cases this would be a difficult task, so that such a choice of axes has restricted application.

x, y, z Axes Having Motion $\mathbf{\Omega} = \boldsymbol{\omega}$. The *x, y, z* axes can be chosen such that they are *fixed in and move with the body*. The moments and products of inertia of the body relative to these axes will then be *constant* during the motion. Since $\mathbf{\Omega} = \boldsymbol{\omega}$, Eqs. 21–22 become

$$\begin{aligned} \Sigma \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_O \\ \Sigma \mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_G \end{aligned} \tag{21–23}$$

We can express each of these vector equations as three scalar equations using Eqs. 21–10. Neglecting the subscripts *O* and *G* yields

$$\begin{aligned} \Sigma M_x &= I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z - I_{xy}(\dot{\omega}_y - \omega_z\omega_x) \\ &\qquad - I_{yz}(\omega_y^2 - \omega_z^2) - I_{zx}(\dot{\omega}_z + \omega_x\omega_y) \\ \Sigma M_y &= I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) \\ &\qquad - I_{zx}(\omega_z^2 - \omega_x^2) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) \\ \Sigma M_z &= I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y - I_{zx}(\dot{\omega}_x - \omega_y\omega_z) \\ &\qquad - I_{xy}(\omega_x^2 - \omega_y^2) - I_{yz}(\dot{\omega}_y + \omega_z\omega_x) \end{aligned} \tag{21–24}$$

If the *x, y, z* axes are chosen as *principal axes of inertia*, the products of inertia are zero, $I_{xx} = I_x$, etc., and the above equations become

$$\boxed{\begin{aligned} \Sigma M_x &= I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z \\ \Sigma M_y &= I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x \\ \Sigma M_z &= I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y \end{aligned}} \tag{21–25}$$

This set of equations is known historically as the *Euler equations of motion*, named after the Swiss mathematician Leonhard Euler, who first developed them. They apply *only* for moments summed about either point *O* or *G*.

When applying these equations it should be realized that $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ represent the time derivatives of the magnitudes of the *x, y, z* components of $\boldsymbol{\omega}$ as observed from *x, y, z*. To determine these components, it is first necessary to find ω_x, ω_y, ω_z when the *x, y, z* axes are oriented in a *general position* and *then* take the time derivative of the magnitude of these components, i.e., $(\dot{\boldsymbol{\omega}})_{xyz}$. However, since the *x, y, z* axes are rotating at $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then from Eq. 20–6, it should be noted that $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz} + \boldsymbol{\omega} \times \boldsymbol{\omega}$. Since $\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. This important result indicates that the time derivative of $\boldsymbol{\omega}$ with respect to the fixed *X,Y,Z* axes, that is $\dot{\boldsymbol{\omega}}$, can also be used to obtain $(\dot{\boldsymbol{\omega}})_{xyz}$. Generally this is the easiest way to determine the result. See Example 21.5.

x, y, z Axes Having Motion $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$. To simplify the calculations for the time derivative of $\boldsymbol{\omega}$, it is often convenient to choose the *x, y, z* axes having an angular velocity $\boldsymbol{\Omega}$ which is different from the angular velocity $\boldsymbol{\omega}$ of the body. This is particularly suitable for the analysis of spinning tops and gyroscopes which are *symmetrical* about their spinning axes.* When this is the case, the moments and products of inertia remain constant about the axis of spin.

Equations 21–22 are applicable for such a set of axes. Each of these two vector equations can be reduced to a set of three scalar equations which are derived in a manner similar to Eqs. 21–25,† i.e.,

$$\begin{aligned} \Sigma M_x &= I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z \\ \Sigma M_y &= I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x \\ \Sigma M_z &= I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y \end{aligned} \qquad (21\text{–}26)$$

Here Ω_x, Ω_y, Ω_z represent the *x, y, z* components of $\boldsymbol{\Omega}$, measured from the inertial frame of reference, and $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ must be determined relative to the *x, y, z* axes that have the rotation $\boldsymbol{\Omega}$. See Example 21.6.

Any one of these sets of moment equations, Eqs. 21–24, 21–25, or 21–26, represents a series of three first-order nonlinear differential equations. These equations are "coupled," since the angular-velocity components are present in all the terms. Success in determining the solution for a particular problem therefore depends upon what is unknown in these equations. Difficulty certainly arises when one attempts to solve for the unknown components of $\boldsymbol{\omega}$ when the external moments are functions of time. Further complications can arise if the moment equations are coupled to the three scalar equations of translational motion, Eqs. 21–19. This can happen because of the existence of kinematic constraints which relate the rotation of the body to the translation of its mass center, as in the case of a hoop which rolls

21

*A detailed discussion of such devices is given in Sec. 21.5.

†See Prob. 21–42.

without slipping. Problems that require the simultaneous solution of differential equations are generally solved using numerical methods with the aid of a computer. In many engineering problems, however, we are given information about the motion of the body and are required to determine the applied moments acting on the body. Most of these problems have direct solutions, so that there is no need to resort to computer techniques.

Procedure for Analysis

Problems involving the three-dimensional motion of a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Draw a *free-body diagram* of the body at the instant considered and specify the x, y, z coordinate system. The origin of this reference must be located either at the body's mass center G, or at point O, considered fixed in an inertial reference frame and located either in the body or on a massless extension of the body.
- Unknown reactive force components can be shown having a positive sense of direction.
- Depending on the nature of the problem, decide what type of rotational motion $\mathbf{\Omega}$ the x, y, z coordinate system should have, i.e., $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{\Omega} = \boldsymbol{\omega}$, or $\mathbf{\Omega} \neq \boldsymbol{\omega}$. When choosing, keep in mind that the moment equations are simplified when the axes move in such a manner that they represent principal axes of inertia for the body at all times.
- Compute the necessary moments and products of inertia for the body relative to the x, y, z axes.

Kinematics.

- Determine the x, y, z components of the body's angular velocity and find the time derivatives of $\boldsymbol{\omega}$.
- Note that if $\mathbf{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. Therefore we can either find the time derivative of $\boldsymbol{\omega}$ with respect to the X, Y, Z axes, $\dot{\boldsymbol{\omega}}$, and then determine its components $\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z$, or we can find the components of $\boldsymbol{\omega}$ along the x, y, z axes, when the axes are oriented in a general position, and then take the time derivative of the magnitudes of these components, $(\dot{\boldsymbol{\omega}})_{xyz}$.

Equations of Motion.

- Apply either the two vector equations 21–18 and 21–22 or the six scalar component equations appropriate for the x, y, z coordinate axes chosen for the problem.

EXAMPLE 21.4

The gear shown in Fig. 21–12*a* has a mass of 10 kg and is mounted at an angle of 10° with the rotating shaft having negligible mass. If $I_z = 0.1\ \text{kg}\cdot\text{m}^2$, $I_x = I_y = 0.05\ \text{kg}\cdot\text{m}^2$, and the shaft is rotating with a constant angular velocity of $\omega = 30$ rad/s, determine the components of reaction that the thrust bearing A and journal bearing B exert on the shaft at the instant shown.

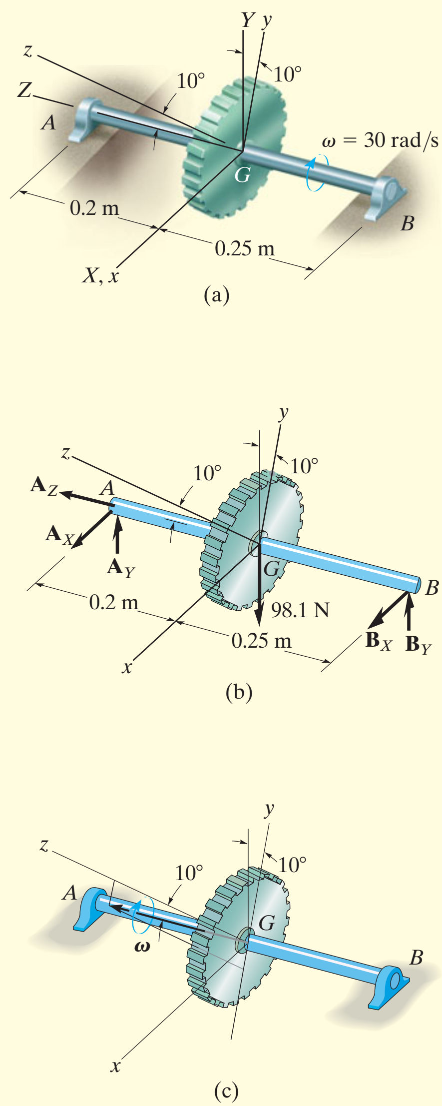


Fig. 21–12

SOLUTION

Free-Body Diagram. Fig. 21–12*b*. The origin of the x, y, z coordinate system is located at the gear's center of mass G, which is also a fixed point. The axes are fixed in and rotate with the gear so that these axes will then always represent the principal axes of inertia for the gear. Hence $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

Kinematics. As shown in Fig. 21–12*c*, the angular velocity $\boldsymbol{\omega}$ of the gear is constant in magnitude and is always directed along the axis of the shaft AB. Since this vector is measured from the X, Y, Z inertial frame of reference, for any position of the x, y, z axes,

$$\omega_x = 0 \qquad \omega_y = -30 \sin 10^\circ \qquad \omega_z = 30 \cos 10^\circ$$

These components remain constant for any general orientation of the x, y, z axes, and so $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Also note that since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. Therefore, we can find these time derivatives relative to the X, Y, Z axes. In this regard $\boldsymbol{\omega}$ has a constant magnitude and direction $(+Z)$ since $\dot{\boldsymbol{\omega}} = \mathbf{0}$, and so $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Furthermore, since G is a fixed point, $(a_G)_x = (a_G)_y = (a_G)_z = 0$.

Equations of Motion. Applying Eqs. 21–25 $(\boldsymbol{\Omega} = \boldsymbol{\omega})$ yields

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$

$$-(A_Y)(0.2) + (B_Y)(0.25) = 0 - (0.05 - 0.1)(-30 \sin 10^\circ)(30 \cos 10^\circ)$$

$$-0.2A_Y + 0.25B_Y = -7.70 \qquad (1)$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$

$$A_X(0.2) \cos 10^\circ - B_X(0.25) \cos 10^\circ = 0 - 0$$

$$A_X = 1.25B_X \qquad (2)$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$A_X(0.2) \sin 10^\circ - B_X(0.25) \sin 10^\circ = 0 - 0$$

$$A_X = 1.25B_X \text{ (check)}$$

Applying Eqs. 21–19, we have

$$\Sigma F_X = m(a_G)_X; \qquad A_X + B_X = 0 \qquad (3)$$

$$\Sigma F_Y = m(a_G)_Y; \qquad A_Y + B_Y - 98.1 = 0 \qquad (4)$$

$$\Sigma F_Z = m(a_G)_Z; \qquad A_Z = 0 \qquad \textit{Ans.}$$

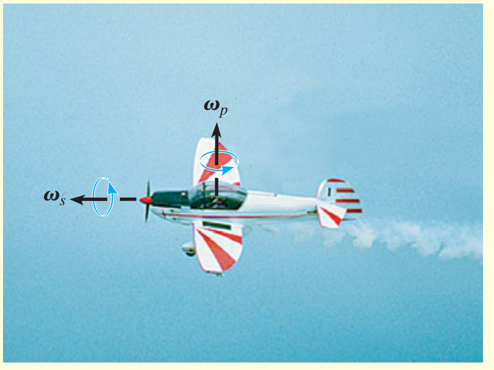
Solving Eqs. 1 through Eqs. 4 simultaneously gives

$$A_X = B_X = 0 \qquad A_Y = 71.6\ \text{N} \qquad B_Y = 26.5\ \text{N} \qquad \textit{Ans.}$$

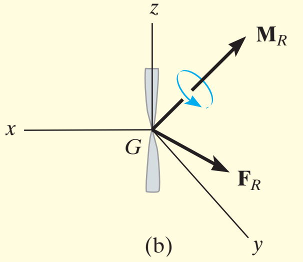
21

EXAMPLE 21.5

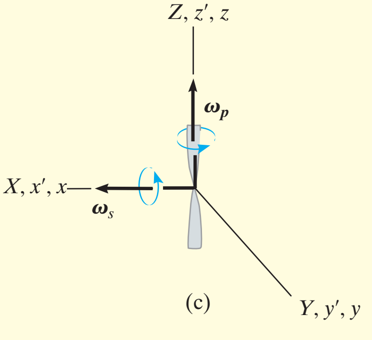
The airplane shown in Fig. 21–13*a* is in the process of making a steady *horizontal* turn at the rate of ω_p. During this motion, the propeller is spinning at the rate of ω_s. If the propeller has two blades, determine the moments which the propeller shaft exerts on the propeller at the instant the blades are in the vertical position. For simplicity, assume the blades to be a uniform slender bar having a moment of inertia I about an axis perpendicular to the blades passing through the center of the bar, and having zero moment of inertia about a longitudinal axis.



(a)



(b)



(c)

Fig. 21–13

SOLUTION

Free-Body Diagram. Fig. 21–13*b*. The reactions of the connecting shaft on the propeller are indicated by the resultants $\mathbf{F}_R$ and $\mathbf{M}_R$. (The propeller's weight is assumed to be negligible.) The *x, y, z* axes will be taken fixed to the propeller, since these axes always represent the principal axes of inertia for the propeller. Thus, $\boldsymbol{\Omega} = \boldsymbol{\omega}$. The moments of inertia I_x and I_y are equal $(I_x = I_y = I)$ and $I_z = 0$.

Kinematics. The angular velocity of the propeller observed from the *X, Y, Z* axes, coincident with the *x, y, z* axes, Fig. 21–13*c*, is $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p = \omega_s\mathbf{i} + \omega_p\mathbf{k}$, so that the *x, y, z* components of $\boldsymbol{\omega}$ are

$$\omega_x = \omega_s \qquad \omega_y = 0 \qquad \omega_z = \omega_p$$

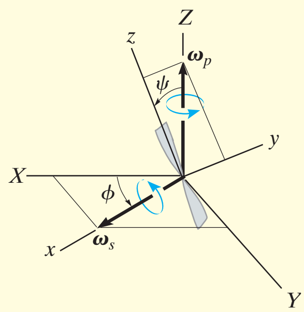
Since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. To find $\dot{\boldsymbol{\omega}}$, which is the time derivative with respect to the fixed *X, Y, Z* axes, we can use Eq. 20–6 since $\boldsymbol{\omega}$ changes direction relative to *X, Y, Z*. The time rate of change of each of these components $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_p$ relative to the *X, Y, Z* axes can be obtained by introducing a third coordinate system x', y', z', which has an angular velocity $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$ and is coincident with the *X, Y, Z* axes at the instant shown. Thus

21

$$\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{x'y'z'} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}$$
$$= (\dot{\boldsymbol{\omega}}_s)_{x'y'z'} + (\dot{\boldsymbol{\omega}}_p)_{x'y'z'} + \boldsymbol{\omega}_p \times (\boldsymbol{\omega}_s + \boldsymbol{\omega}_p)$$
$$= \mathbf{0} + \mathbf{0} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_s + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_p$$
$$= \mathbf{0} + \mathbf{0} + \omega_p \mathbf{k} \times \omega_s \mathbf{i} + \mathbf{0} = \omega_p \omega_s \mathbf{j}$$

Since the X, Y, Z axes are coincident with the x, y, z axes at the instant shown, the components of $\dot{\boldsymbol{\omega}}$ along x, y, z are therefore

$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p \omega_s \qquad \dot{\omega}_z = 0$$



(d)

These same results can also be determined by direct calculation of $(\dot{\boldsymbol{\omega}})_{xyz}$; however, this will involve a bit more work. To do this, it will be necessary to view the propeller (or the x, y, z axes) in some *general position* such as shown in Fig. 21–13*d*. Here the plane has turned through an angle ϕ (phi) and the propeller has turned through an angle ψ (psi) relative to the plane. Notice that $\boldsymbol{\omega}_p$ is always directed along the fixed Z axis and $\boldsymbol{\omega}_s$ follows the x axis. Thus the general components of $\boldsymbol{\omega}$ are

$$\omega_x = \omega_s \qquad \omega_y = \omega_p \sin \psi \qquad \omega_z = \omega_p \cos \psi$$

Since ω_s and ω_p are constant, the time derivatives of these components become

$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p \cos \psi \, \dot{\psi} \qquad \omega_z = -\omega_p \sin \psi \, \dot{\psi}$$

But $\phi = \psi = 0^\circ$ and $\dot{\psi} = \omega_s$ at the instant considered. Thus,

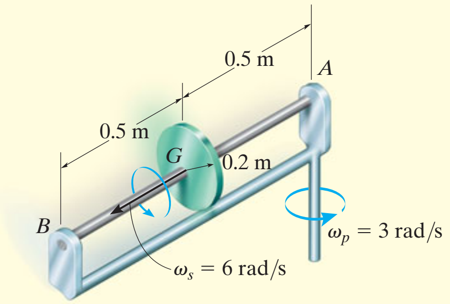
$$\omega_x = \omega_s \qquad \omega_y = 0 \qquad \omega_z = \omega_p$$
$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p \omega_s \qquad \dot{\omega}_z = 0$$

which are the same results as those obtained previously.

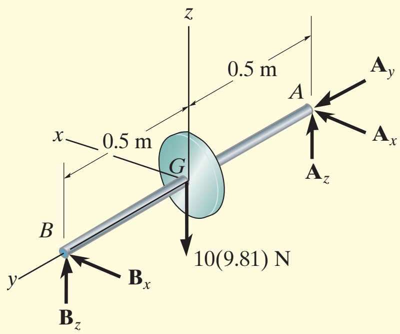
Equations of Motion. Using Eqs. 21–25, we have

$$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z = I(0) - (I - 0)(0)\omega_p$$
$$M_x = 0 \qquad \textit{Ans.}$$
$$\Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x)\omega_z \omega_x = I(\omega_p \omega_s) - (0 - I)\omega_p \omega_s$$
$$M_y = 2I\omega_p \omega_s \qquad \textit{Ans.}$$
$$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y = 0(0) - (I - I)\omega_s(0)$$
$$M_z = 0 \qquad \textit{Ans.}$$

EXAMPLE 21.6



(a)



(b)

Fig. 21–14

The 10-kg flywheel (or thin disk) shown in Fig. 21–14*a* rotates (spins) about the shaft at a constant angular velocity of $\omega_s = 6$ rad/s. At the same time, the shaft rotates (precessing) about the bearing at A with an angular velocity of $\omega_p = 3$ rad/s. If A is a thrust bearing and B is a journal bearing, determine the components of force reaction at each of these supports due to the motion.

SOLUTION I

Free-Body Diagram. Fig. 21–14*b*. The origin of the x, y, z coordinate system is located at the center of mass G of the flywheel. Here we will let these coordinates have an angular velocity of $\boldsymbol{\Omega} = \boldsymbol{\omega}_p = \{3\mathbf{k}\}$ rad/s. Although the wheel spins relative to these axes, the moments of inertia *remain constant*,* i.e.,

$$I_x = I_z = \tfrac{1}{4}(10\text{ kg})(0.2\text{ m})^2 = 0.1\text{ kg}\cdot\text{m}^2$$
$$I_y = \tfrac{1}{2}(10\text{ kg})(0.2\text{ m})^2 = 0.2\text{ kg}\cdot\text{m}^2$$

Kinematics. From the coincident inertial X, Y, Z frame of reference, Fig. 21–14*c*, the flywheel has an angular velocity of $\boldsymbol{\omega} = \{6\mathbf{j} + 3\mathbf{k}\}$ rad/s, so that

$$\omega_x = 0 \quad \omega_y = 6\text{ rad/s} \quad \omega_z = 3\text{ rad/s}$$

The time derivative of $\boldsymbol{\omega}$ must be determined relative to the x, y, z axes. In this case both $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_s$ do not change their magnitude or direction, and so

$$\dot{\omega}_x = 0 \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

Equations of Motion. Applying Eqs. 21–26 ($\boldsymbol{\Omega} \neq \boldsymbol{\omega}$) yields

$$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z$$
$$-A_z(0.5) + B_z(0.5) = 0 - (0.2)(3)(6) + 0 = -3.6$$
$$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x$$
$$0 = 0 - 0 + 0$$
$$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y$$
$$A_x(0.5) - B_x(0.5) = 0 - 0 + 0$$

* This would not be true for the propeller in Example 21.5.

Applying Eqs. 21–19, we have

$$\Sigma F_X = m(a_G)_X; \qquad A_x + B_x = 0$$

$$\Sigma F_Y = m(a_G)_Y; \qquad A_y = -10(0.5)(3)^2$$

$$\Sigma F_Z = m(a_G)_Z; \qquad A_z + B_z - 10(9.81) = 0$$

Solving these equations, we obtain

$$A_x = 0 \quad A_y = -45.0 \text{ N} \quad A_z = 52.6 \text{ N} \qquad \textit{Ans.}$$
$$B_x = 0 \quad B_z = 45.4 \text{ N} \qquad \textit{Ans.}$$

NOTE: If the precession $\boldsymbol{\omega}_p$ had not occurred, the z component of force at A and B would be equal to 49.05 N. In this case, however, the difference in these components is caused by the "gyroscopic moment" created whenever a spinning body precesses about another axis. We will study this effect in detail in the next section.

SOLUTION II

This example can also be solved using Euler's equations of motion, Eqs. 21–25. In this case $\boldsymbol{\Omega} = \boldsymbol{\omega} = \{6\mathbf{j} + 3\mathbf{k}\}$ rad/s, and the time derivative $(\dot{\boldsymbol{\omega}})_{xyz}$ can be conveniently obtained with reference to the fixed X, Y, Z axes since $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. This calculation can be performed by choosing x', y', z' axes to have an angular velocity of $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$, Fig. 21–14*c*, so that

$$\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{x'y'z'} + \boldsymbol{\omega}_p \times \boldsymbol{\omega} = \mathbf{0} + 3\mathbf{k} \times (6\mathbf{j} + 3\mathbf{k}) = \{-18\mathbf{i}\} \text{ rad/s}^2$$

$$\dot{\omega}_x = -18 \text{ rad/s} \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

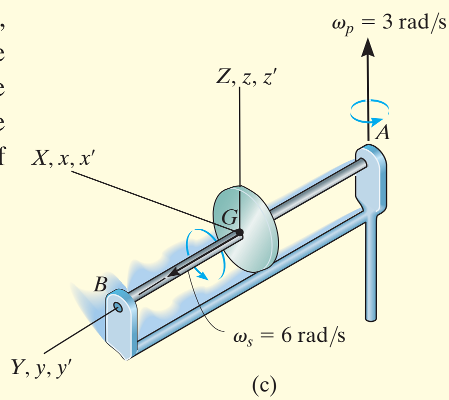


Fig. 21–14

The moment equations then become

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$

$$-A_z(0.5) + B_z(0.5) = 0.1(-18) - (0.2 - 0.1)(6)(3) = -3.6$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$

$$0 = 0 - 0$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$A_x(0.5) - B_x(0.5) = 0 - 0$$

The solution then proceeds as before.

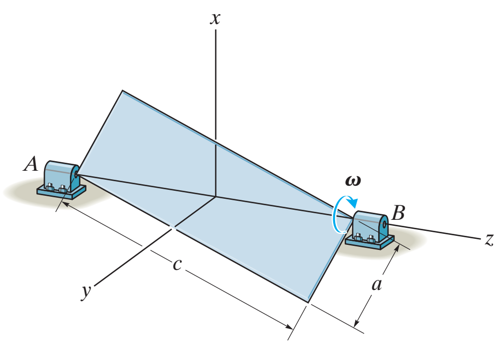
PROBLEMS

***21–40.** Derive the scalar form of the rotational equation of motion about the x axis if $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *not constant* with respect to time.

•21–41. Derive the scalar form of the rotational equation of motion about the x axis if $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *constant* with respect to time.

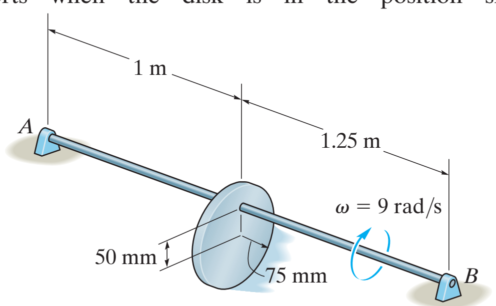
21–42. Derive the Euler equations of motion for $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$, i.e., Eqs. 21–26.

21–43. The uniform rectangular plate has a mass of $m = 2$ kg and is given a rotation of $\omega = 4$ rad/s about its bearings at A and B. If $a = 0.2$ m and $c = 0.3$ m, determine the vertical reactions at A and B at the instant the plate is vertical as shown. Use the x, y, z axes shown and note that $I_{zx} = -\left(\frac{mac}{12}\right)\left(\frac{c^2 - a^2}{c^2 + a^2}\right)$.



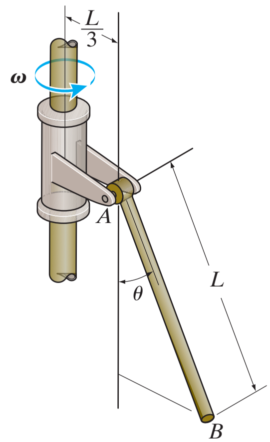
Prob. 21–43

***21–44.** The disk, having a mass of 3 kg, is mounted eccentrically on shaft AB. If the shaft is rotating at a constant rate of 9 rad/s, determine the reactions at the journal bearing supports when the disk is in the position shown.



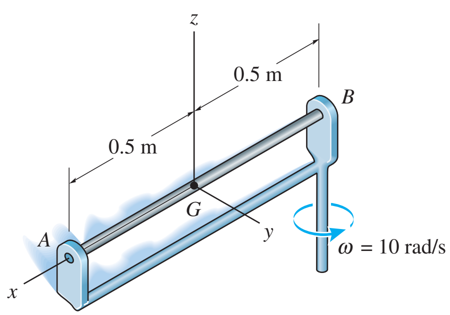
Prob. 21–44

•21–45. The slender rod AB has a mass m and it is connected to the bracket by a smooth pin at A. The bracket is rigidly attached to the shaft. Determine the required constant angular velocity of $\boldsymbol{\omega}$ of the shaft, in order for the rod to make an angle of θ with the vertical.



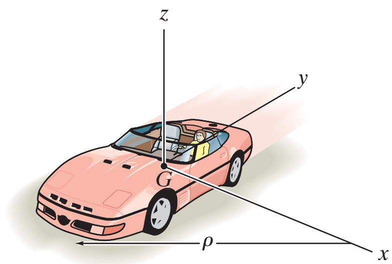
Prob. 21–45

21–46. The 5-kg rod AB is supported by a rotating arm. The support at A is a journal bearing, which develops reactions normal to the rod. The support at B is a thrust bearing, which develops reactions both normal to the rod and along the axis of the rod. Neglecting friction, determine the x, y, z components of reaction at these supports when the frame rotates with a constant angular velocity of $\omega = 10$ rad/s.



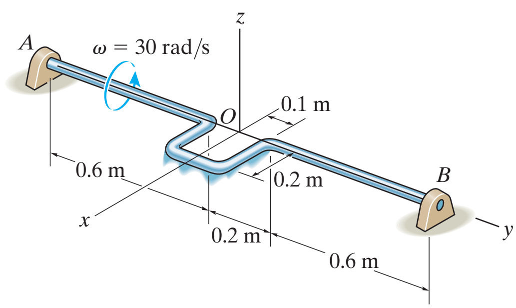
Prob. 21–46

21–47. The car travels around the curved road of radius ρ such that its mass center has a constant speed v_G. Write the equations of rotational motion with respect to the x, y, z axes. Assume that the car's six moments and products of inertia with respect to these axes are known.



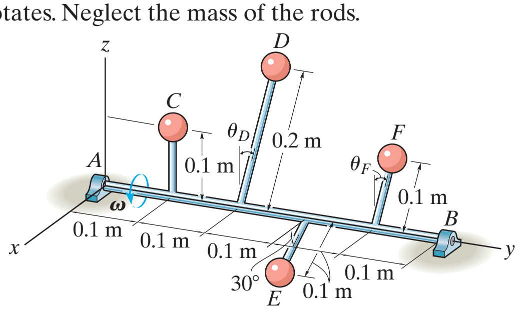
Prob. 21–47

***21–48.** The shaft is constructed from a rod which has a mass per unit length of 2 kg/m. Determine the x, y, z components of reaction at the bearings A and B if at the instant shown the shaft spins freely and has an angular velocity of $\omega = 30$ rad/s. What is the angular acceleration of the shaft at this instant? Bearing A can support a component of force in the y direction, whereas bearing B cannot.



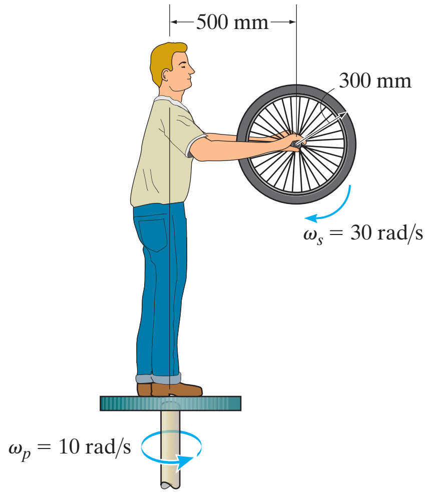
Prob. 21–48

•**21–49.** Four spheres are connected to shaft AB. If $m_C = 1$ kg and $m_E = 2$ kg, determine the mass of spheres D and F and the angles of the rods, θ_D and θ_F, so that the shaft is dynamically balanced, that is, so that the bearings at A and B exert only vertical reactions on the shaft as it rotates. Neglect the mass of the rods.



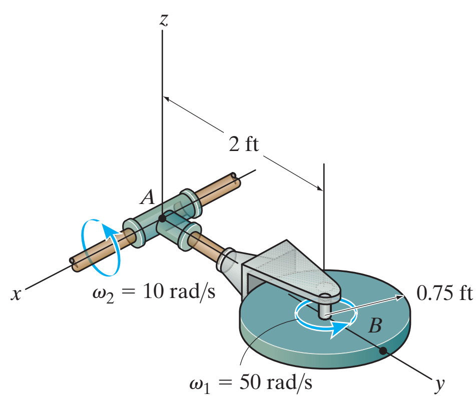
Prob. 21–49

21–50. A man stands on a turntable that rotates about a vertical axis with a constant angular velocity of $\omega_p = 10$ rad/s. If the wheel that he holds spins with a constant angular speed of $\omega_s = 30$ rad/s, determine the magnitude of moment that he must exert on the wheel to hold it in the position shown. Consider the wheel as a thin circular hoop (ring) having a mass of 3 kg and a mean radius of 300 mm.



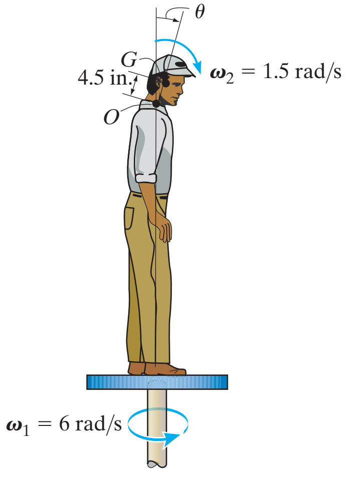
Prob. 21–50

21–51. The 50-lb disk spins with a constant angular rate of $\omega_1 = 50$ rad/s about its axle. Simultaneously, the shaft rotates with a constant angular rate of $\omega_2 = 10$ rad/s. Determine the x, y, z components of the moment developed in the arm at A at the instant shown. Neglect the weight of arm AB.



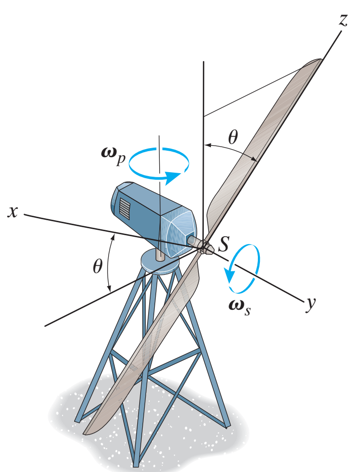
Prob. 21–51

***21–52.** The man stands on a turntable that rotates about a vertical axis with a constant angular velocity of $\omega_1 = 6$ rad/s. If he tilts his head forward at a constant angular velocity of $\omega_2 = 1.5$ rad/s about point O, determine the magnitude of the moment that must be resisted by his neck at O at the instant $\theta = 30°$. Assume that his head can be considered as a uniform 10-lb sphere, having a radius of 4.5 in. and center of gravity located at G, and point O is on the surface of the sphere.



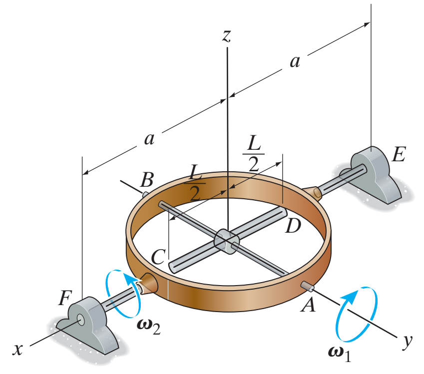
Prob. 21–52

•21–53. The blades of a wind turbine spin about the shaft S with a constant angular speed of ω_s, while the frame precesses about the vertical axis with a constant angular speed of ω_p. Determine the x, y, and z components of moment that the shaft exerts on the blades as a function of θ. Consider each blade as a slender rod of mass m and length l.



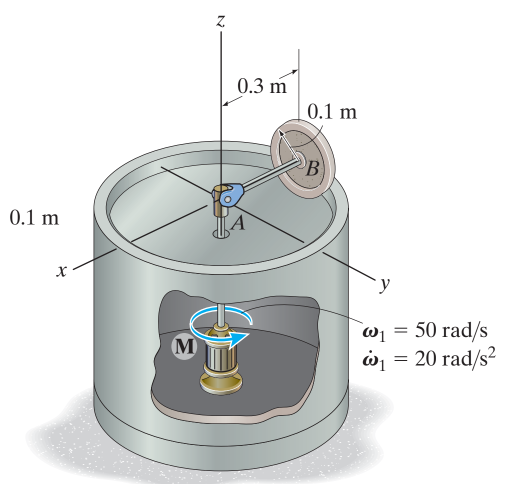
Prob. 21–53

21–54. Rod CD of mass m and length L is rotating with a constant angular rate of ω_1 about axle AB, while shaft EF rotates with a constant angular rate of ω_2. Determine the X, Y, and Z components of reaction at thrust bearing E and journal bearing F at the instant shown. Neglect the mass of the other members.



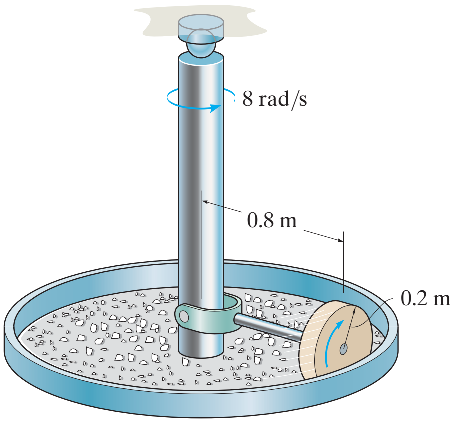
Prob. 21–54

21–55. If shaft AB is driven by the motor with an angular velocity of $\omega_1 = 50$ rad/s and angular acceleration of $\dot{\omega}_1 = 20$ rad/s^2 at the instant shown, and the 10-kg wheel rolls without slipping, determine the frictional force and the normal reaction on the wheel, and the moment **M** that must be supplied by the motor at this instant. Assume that the wheel is a uniform circular disk.



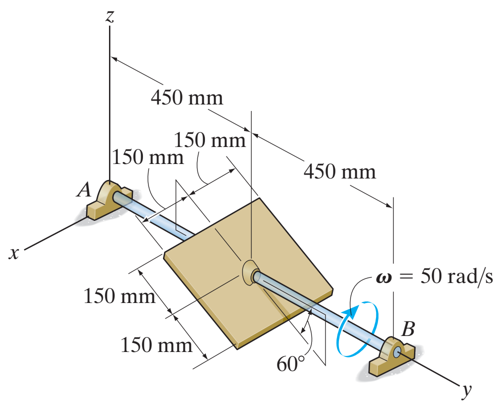
Prob. 21–55

***21–56.** A stone crusher consists of a large thin disk which is pin connected to a horizontal axle. If the axle rotates at a constant rate of 8 rad/s, determine the normal force which the disk exerts on the stones. Assume that the disk rolls without slipping and has a mass of 25 kg. Neglect the mass of the axle.



Prob. 21–56

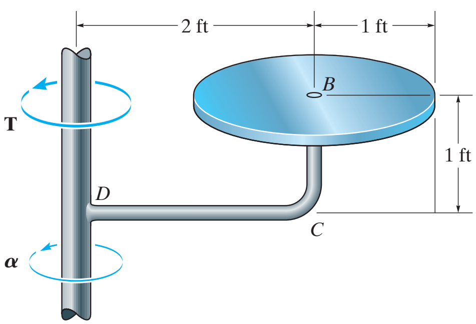
21–59. If shaft AB rotates with a constant angular velocity of $\omega = 50$ rad/s, determine the X, Y, Z components of reaction at journal bearing A and thrust bearing B at the instant shown. The thin plate has a mass of 10 kg. Neglect the mass of shaft AB.



Prob. 21–59

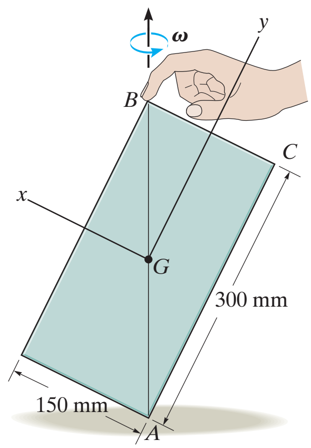
•21–57. The 25-lb disk is *fixed* to rod BCD, which has negligible mass. Determine the torque **T** which must be applied to the vertical shaft so that the shaft has an angular acceleration of $\alpha = 6$ rad/s^2. The shaft is free to turn in its bearings.

21–58. Solve Prob. 21–57, assuming rod BCD has a weight per unit length of 2 lb/ft.



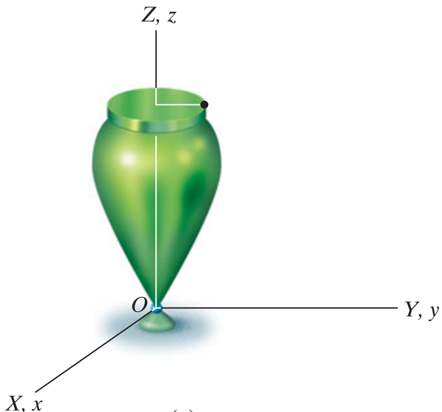
Probs. 21–57/58

***21–60.** A thin uniform plate having a mass of 0.4 kg spins with a constant angular velocity $\boldsymbol{\omega}$ about its diagonal AB. If the person holding the corner of the plate at B releases his finger, the plate will fall downward on its side AC. Determine the necessary couple moment **M** which if applied to the plate would prevent this from happening.

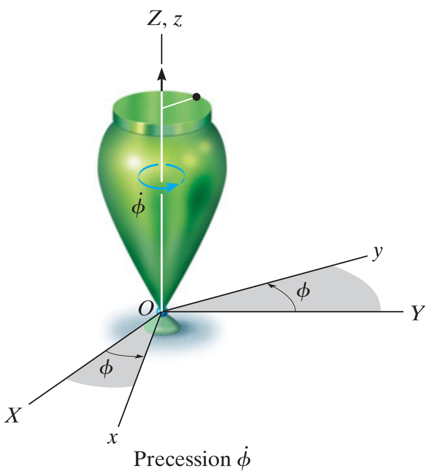


Prob. 21–60

21



(a)



(b)

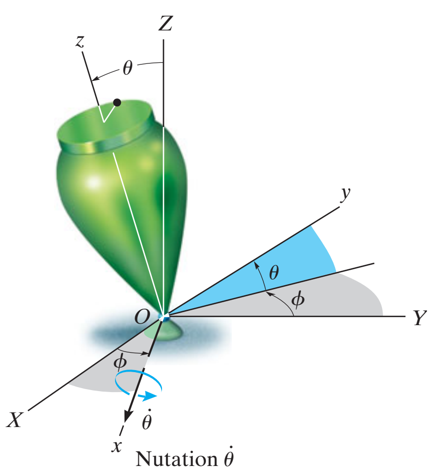
*21.5 Gyroscopic Motion

In this section we will develop the equations defining the motion of a body (top) which is symmetrical with respect to an axis and rotating about a fixed point. These equations also apply to the motion of a particularly interesting device, the gyroscope.

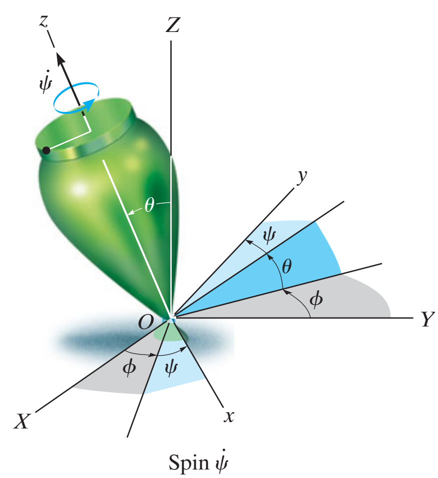
The body's motion will be analyzed using *Euler angles* ϕ, θ, ψ (phi, theta, psi). To illustrate how they define the position of a body, consider the top shown in Fig. 21–15*a*. To define its final position, Fig. 21–15*d*, a second set of *x, y, z* axes is fixed in the top. Starting with the *X, Y, Z* and *x, y, z* axes in coincidence, Fig. 21–15*a*, the final position of the top can be determined using the following three steps:

1. Rotate the top about the Z (or z) axis through an angle ϕ $(0 \leq \phi < 2\pi)$, Fig. 21–15*b*.
2. Rotate the top about the x axis through an angle θ $(0 \leq \theta \leq \pi)$, Fig. 21–15*c*.
3. Rotate the top about the z axis through an angle ψ $(0 \leq \psi < 2\pi)$ to obtain the final position, Fig. 20–15*d*.

The sequence of these three angles, ϕ, θ, then ψ, must be maintained, since finite rotations are *not vectors* (see Fig. 20–1). Although this is the case, the differential rotations $d\boldsymbol{\phi}$, $d\boldsymbol{\theta}$, and $d\boldsymbol{\psi}$ are vectors, and thus the angular velocity $\boldsymbol{\omega}$ of the top can be expressed in terms of the time derivatives of the Euler angles. The angular-velocity components $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are known as the *precession, nutation*, and *spin*, respectively.



(c)



(d)

Fig. 21–15

Their positive directions are shown in Fig. 21–16. It is seen that these vectors are not all perpendicular to one another; however, $\boldsymbol{\omega}$ of the top can still be expressed in terms of these three components.

Since the body (top) is symmetric with respect to the z or spin axis, there is no need to attach the x, y, z axes to the top since the inertial properties of the top will remain constant with respect to this frame during the motion. Therefore $\boldsymbol{\Omega} = \boldsymbol{\omega}_p + \boldsymbol{\omega}_n$, Fig. 21–16. Hence, the angular velocity of the body is

$$\begin{aligned}\boldsymbol{\omega} &= \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \\ &= \dot{\theta}\mathbf{i} + (\dot{\phi}\sin\theta)\mathbf{j} + (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{k} \qquad (21\text{–}27)\end{aligned}$$

And the angular velocity of the axes is

$$\begin{aligned}\boldsymbol{\Omega} &= \Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k} \\ &= \dot{\theta}\mathbf{i} + (\dot{\phi}\sin\theta)\mathbf{j} + (\dot{\phi}\cos\theta)\mathbf{k} \qquad (21\text{–}28)\end{aligned}$$

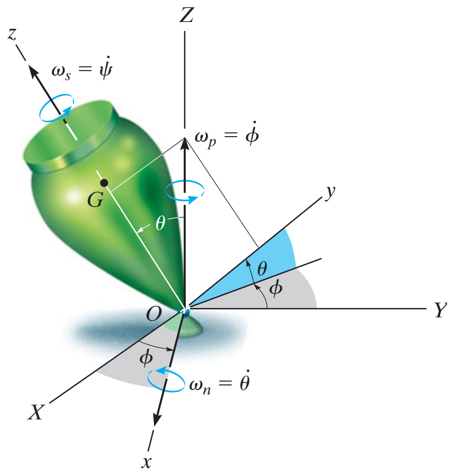


Fig. 21–16

Have the x, y, z axes represent principal axes of inertia for the top, and so the moments of inertia will be represented as $I_{xx} = I_{yy} = I$ and $I_{zz} = I_z$. Since $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$, Eqs. 21–26 are used to establish the rotational equations of motion. Substituting into these equations the respective angular-velocity components defined by Eqs. 21–27 and 21–28, their corresponding time derivatives, and the moment of inertia components, yields

$$\begin{aligned}\Sigma M_x &= I(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta) + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi}) \\ \Sigma M_y &= I(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I_z\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) \qquad (21\text{–}29) \\ \Sigma M_z &= I_z(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta)\end{aligned}$$

Each moment summation applies only at the fixed point O or the center of mass G of the body. Since the equations represent a coupled set of nonlinear second-order differential equations, in general a closed-form solution may not be obtained. Instead, the Euler angles ϕ, θ, and ψ may be obtained graphically as functions of time using numerical analysis and computer techniques.

A special case, however, does exist for which simplification of Eqs. 21–29 is possible. Commonly referred to as *steady precession*, it occurs when the nutation angle θ, precession $\dot{\phi}$, and spin $\dot{\psi}$ all remain *constant*. Equations 21–29 then reduce to the form

$$\boxed{\Sigma M_x = -I\dot{\phi}^2\sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})} \qquad (21\text{–}30)$$

$$\Sigma M_y = 0$$

$$\Sigma M_z = 0$$

Equation 21–30 can be further simplified by noting that, from Eq. 21–27, $\omega_z = \dot{\phi}\cos\theta + \dot{\psi}$, so that

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\,(\sin\theta)\omega_z$$

or

$$\boxed{\Sigma M_x = \dot{\phi}\sin\theta(I_z\omega_z - I\dot{\phi}\cos\theta)} \qquad (21\text{–}31)$$

It is interesting to note what effects the spin $\dot{\psi}$ has on the moment about the x axis. To show this, consider the spinning rotor in Fig. 21–17. Here $\theta = 90°$, in which case Eq. 21–30 reduces to the form

$$\Sigma M_x = I_z\dot{\phi}\dot{\psi}$$

or

$$\boxed{\Sigma M_x = I_z\Omega_y\omega_z} \qquad (21\text{–}32)$$

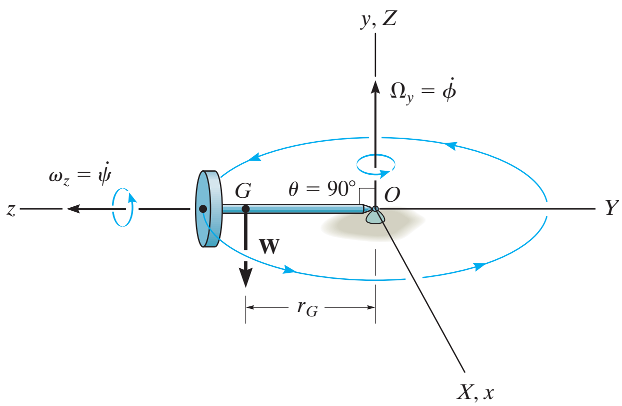


Fig. 21–17

From the figure it can be seen that $\mathbf{\Omega}_y$ and $\boldsymbol{\omega}_z$ act along their respective *positive axes* and therefore are mutually perpendicular. Instinctively, one would expect the rotor to fall down under the influence of gravity! However, this is not the case at all, provided the product $I_z\Omega_y\omega_z$ is correctly chosen to counterbalance the moment $\Sigma M_x = Wr_G$ of the rotor's weight about O. This unusual phenomenon of rigid-body motion is often referred to as the *gyroscopic effect*.

Perhaps a more intriguing demonstration of the gyroscopic effect comes from studying the action of a *gyroscope*, frequently referred to as a *gyro*. A gyro is a rotor which spins at a very high rate about its axis of symmetry. This rate of spin is considerably greater than its precessional rate of rotation about the vertical axis. Hence, for all practical purposes, the angular momentum of the gyro can be assumed directed along its axis of spin. Thus, for the gyro rotor shown in Fig. 21–18, $\omega_z \gg \Omega_y$, and the magnitude of the angular momentum about point O, as determined from Eqs. 21–11, reduces to the form $H_O = I_z\omega_z$. Since both the magnitude and direction of $\mathbf{H}_O$ are constant as observed from x, y, z, direct application of Eq. 21–22 yields

$$\boxed{\Sigma \mathbf{M}_x = \mathbf{\Omega}_y \times \mathbf{H}_O} \tag{21–33}$$

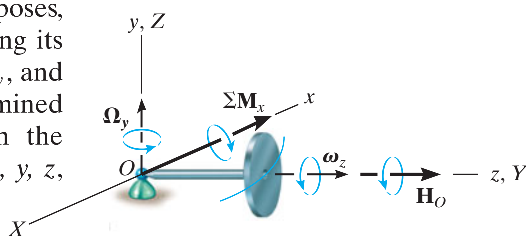


Fig. 21–18

Using the right-hand rule applied to the cross product, it can be seen that $\mathbf{\Omega}_y$ always swings $\mathbf{H}_O$ (or $\boldsymbol{\omega}_z$) toward the sense of $\Sigma\mathbf{M}_x$. In effect, the *change in direction* of the gyro's angular momentum, $d\mathbf{H}_O$, is equivalent to the angular impulse caused by the gyro's weight about O, i.e., $d\mathbf{H}_O = \Sigma\mathbf{M}_x\,dt$, Eq. 21–20. Also, since $H_O = I_z\omega_z$ and $\Sigma\mathbf{M}_x$, $\mathbf{\Omega}_y$, and $\mathbf{H}_O$ are mutually perpendicular, Eq. 21–33 reduces to Eq. 21–32.

When a gyro is mounted in gimbal rings, Fig. 21–19, it becomes *free* of external moments applied to its base. Thus, in theory, its angular momentum $\mathbf{H}$ will never precess but, instead, maintain its same fixed orientation along the axis of spin when the base is rotated. This type of gyroscope is called a *free gyro* and is useful as a gyrocompass when the spin axis of the gyro is directed north. In reality, the gimbal mechanism is never completely free of friction, so such a device is useful only for the local navigation of ships and aircraft. The gyroscopic effect is also useful as a means of stabilizing both the rolling motion of ships at sea and the trajectories of missiles and projectiles. Furthermore, this effect is of significant importance in the design of shafts and bearings for rotors which are subjected to forced precessions.

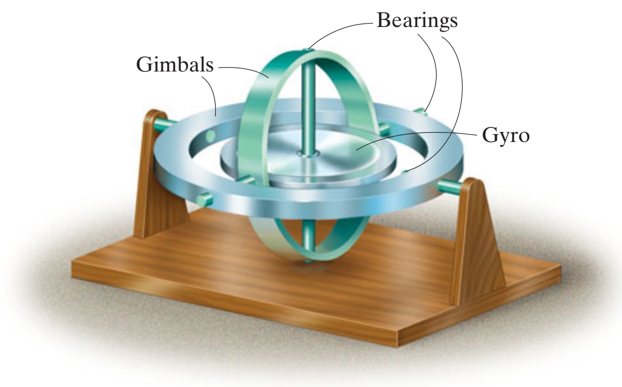
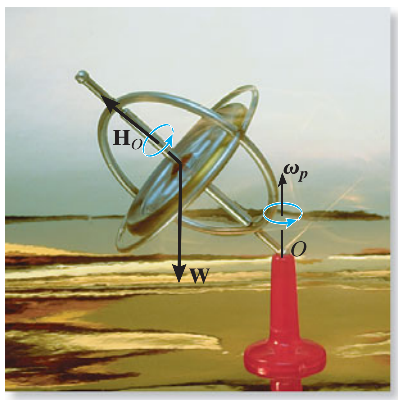
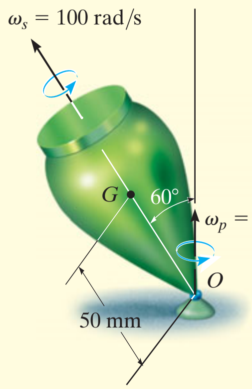


Fig. 21–19



The spinning of the gyro within the frame of this toy gyroscope produces angular momentum $\mathbf{H}_O$, which is changing direction as the frame precesses $\boldsymbol{\omega}_p$ about the vertical axis. The gyroscope will not fall down since the moment of its weight $\mathbf{W}$ about the support is balanced by the change in the direction of $\mathbf{H}_O$.

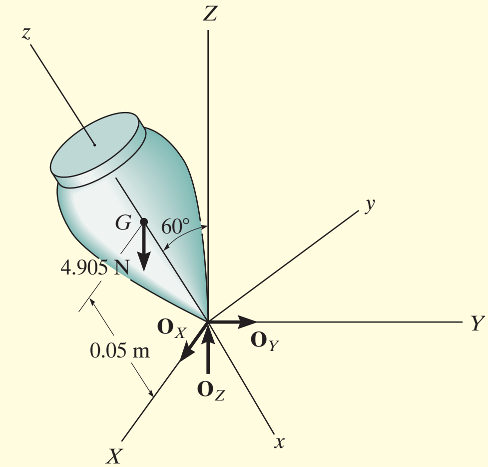
EXAMPLE 21.7



(a)

Fig. 21–20

The top shown in Fig. 21–20*a* has a mass of 0.5 kg and is precessing about the vertical axis at a constant angle of $\theta = 60°$. If it spins with an angular velocity $\omega_s = 100$ rad/s, determine the precession $\boldsymbol{\omega}_p$. Assume that the axial and transverse moments of inertia of the top are $0.45(10^{-3})\ \text{kg}\cdot\text{m}^2$ and $1.20(10^{-3})\ \text{kg}\cdot\text{m}^2$, respectively, measured with respect to the fixed point O.



(b)

SOLUTION

Equation 21–30 will be used for the solution since the motion is *steady precession*. As shown on the free-body diagram, Fig. 21–20*b*, the coordinate axes are established in the usual manner, that is, with the positive z axis in the direction of spin, the positive Z axis in the direction of precession, and the positive x axis in the direction of the moment ΣM_x (refer to Fig. 21–16). Thus,

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$$

$$4.905\ \text{N}(0.05\ \text{m})\sin 60° = -[1.20(10^{-3})\ \text{kg}\cdot\text{m}^2\ \dot{\phi}^2]\sin 60°\cos 60°$$
$$+ [0.45(10^{-3})\ \text{kg}\cdot\text{m}^2]\dot{\phi}\sin 60°(\dot{\phi}\cos 60° + 100\ \text{rad/s})$$

or

$$\dot{\phi}^2 - 120.0\dot{\phi} + 654.0 = 0 \qquad (1)$$

Solving this quadratic equation for the precession gives

$$\dot{\phi} = 114\ \text{rad/s} \qquad \text{(high precession)}$$ *Ans.*

and

$$\dot{\phi} = 5.72\ \text{rad/s} \qquad \text{(low precession)}$$ *Ans.*

NOTE: In reality, low precession of the top would generally be observed, since high precession would require a larger kinetic energy.

21

EXAMPLE 21.8

The 1-kg disk shown in Fig. 21–21*a* spins about its axis with a constant angular velocity $\omega_D = 70$ rad/s. The block at B has a mass of 2 kg, and by adjusting its position s one can change the precession of the disk about its supporting pivot at O while the shaft remains horizontal. Determine the position s that will enable the disk to have a constant precession $\omega_p = 0.5$ rad/s about the pivot. Neglect the weight of the shaft.

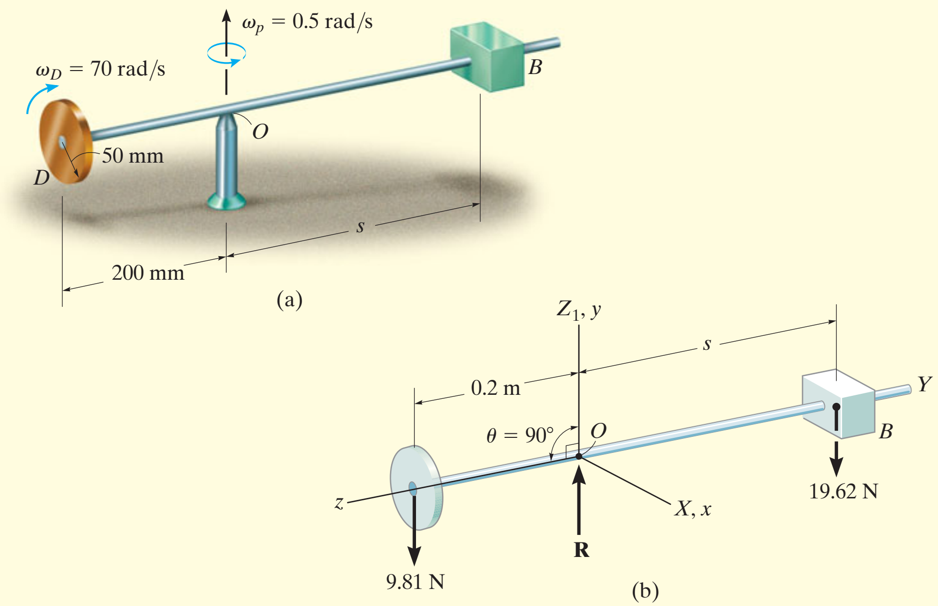


Fig. 21–21

SOLUTION

The free-body diagram of the assembly is shown in Fig. 21–21*b*. The origin for both the x, y, z and X, Y, Z coordinate systems is located at the fixed point O. In the conventional sense, the Z axis is chosen along the axis of precession, and the z axis is along the axis of spin, so that $\theta = 90°$. Since the precession is *steady*, Eq. 21–32 can be used for the solution.

$$\Sigma M_x = I_z \Omega_y \omega_z$$

Substituting the required data gives

$$(98.1 \text{ N})\ (0.2 \text{ m}) - (19.62 \text{ N})s = \left[\tfrac{1}{2}(1 \text{ kg})(0.05 \text{ m})^2\right] 0.5 \text{ rad/s}(-70 \text{ rad/s})$$

$$s = 0.102 \text{ m} = 102 \text{ mm} \qquad \textit{Ans.}$$